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Asymptotic stability of linear delay differential–algebraic equations and numerical methods ☆

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Abstract

In this paper, we consider the asymptotic stability of linear constant coefficient delay differential–algebraic equations and of θ-methods, Runge–Kutta methods and linear multistep methods applied to these systems. © 1997 Published by Elsevier Science B.V.

Keywords: Differential–algebraic equations; Delays; Asymptotic stability

1. Introduction

In recent years, much research has been focused on the numerical solution of systems of differential–algebraic equations (DAEs) [5,10,22]. These systems can be found in a wide variety of scientific and engineering applications, including circuit analysis, computer-aided design and real-time simulation of mechanical (multibody) systems, power systems, chemical process simulation, and optimal control.

During the same period of time much work has also been done in the field of numerical solution of delay differential equations (DDEs) [3,6,11,13–21,26,27]. Delay differential equations arise from, for example, real-time simulation, where time delays can be introduced by the computer time needed to compute an output after the input has been sampled, and where additional delays can be introduced by the operator-in-the-loop [9]. Delays arise also in circuit simulation and power systems, due to, for example, interconnects for computer chips [12] and transmission lines [24], and in chemical process simulation when modeling pipe flows [25].

Delay differential–algebraic equations (DDAEs), which have both delay and algebraic constraints, appear frequently in these fields. However, not much work has been done on numerical methods for DDAEs. In [2,7,8,11], the structure of DDAEs and order and convergence of numerical methods have

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been studied but the asymptotic stability of these systems and numerical methods still remains to be investigated.

The stability of numerical methods for DDEs has been very intensively studied. Different kinds of stability have been defined [27], and stability of RK methods has been studied in [6,13,15,17,19,26], where scalar or systems of DDEs with constant or variable delays are considered. In [4,14,16,20,21], stability of θ-methods is studied for DDEs with different structures. Most of these results are for linear constant coefficient systems with constant delay.

In this paper, we focus on asymptotic stability of numerical methods for linear constant coefficient DDAEs. We first study the asymptotic stability of multistep methods and RK methods for linear constant coefficient DAEs, which helps us understand the DDAE case better. We then give conditions for linear constant coefficient DDAEs to be asymptotically stable, followed by results on the asymptotic stability of θ-methods, multistep methods and RK methods. Stability results for nonlinear DDAEs will be given in a subsequent paper.

2. Asymptotic stability results for DAE

In this section, we consider asymptotic stability of linear constant coefficient DAE and numerical methods.

2.1. Linear constant coefficient DAE

We consider DAEs of the form

$$Ax' + Bx = 0, \tag{2.1}$$

where $A, B \in \mathbb{R}^{n\times n}$ are constant matrices and A is singular. This is the simplest kind of DAE. Solvability, which is essentially the existence and uniqueness of the solution, is given by the following theorem in [5].

Theorem 2.1. *The system* (2.1) *is solvable if and only if the matrix pencil* $\lambda A + B$ *is regular, i.e., not identically singular for any* λ.

We give the definition of asymptotic stability of the solution of (2.1) here.

Definition 2.1. The solution $x(t)$ of (2.1) is said to be asymptotically stable if there exists a constant b such that, for any other solution, $y(t)$ of (2.1) satisfying

$$\left|x(t_0) - y(t_0)\right| < b, \qquad \lim_{t\to\infty} \left|x(t) - y(t)\right| = 0.$$

The following result concerning asymptotic stability of the null solution of (2.1) is given in [23].

Theorem 2.2. *The null solution of DAE system* (2.1) *is asymptotically stable iff the singular values of the matrix pencil* (A, B) *all have negative real part.*

Here, the singular value of the matrix pencil (A, B) is $\lambda \in C$, such that $\det[\lambda A + B] = 0$. Thus for the system (2.1) to be asymptotically stable, B is required to be nonsingular since otherwise $\lambda = 0$ will be a singular value of the matrix pencil (A, B).

To determine the asymptotic stability of a given system, we note that it suffices to consider the case that the matrices A and B can be transformed to triangular matrices simultaneously. To see this, we first give the following theorem from [5].

Theorem 2.3. *Suppose that $\lambda A + B$ is a regular pencil. Then there exist nonsingular matrices P, Q such that*

$$PAQ = \begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix}, \qquad PBQ = \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix},$$

where N is a nilpotent matrix.

By the Schur decomposition theorem, we always can find unitary matrices P_1, Q_1 such that

$$P_1^H C P_1 = \overline{C}, \qquad Q_1^H N Q_1 = \overline{N}$$

are triangular matrices. Because N is nilpotent, $\overline{N}$ is strictly triangular. Thus the nonsingular matrices

$$\begin{pmatrix} P_1 & 0 \\ 0 & Q_1 \end{pmatrix}^H P, \qquad Q \begin{pmatrix} P_1 & 0 \\ 0 & Q_1 \end{pmatrix},$$

will transform the matrices A and B into triangular form.

2.2. Asymptotic stability of linear multistep methods

Asymptotic stability of numerical methods for system (2.1) is similarly defined. Instead of the exact solution, the numerical discrete solution is considered.

Consider the multistep method

$$\sum_{j=0}^{s} \alpha_j x_{n+j} = h \sum_{j=0}^{s} \beta_j f_{n+j}. \tag{2.2}$$

When (2.2) is applied to (2.1), we have

$$\sum_{j=0}^{s} (\alpha_j A x_{n+j} + h\beta_j B x_{n+j}) = 0. \tag{2.3}$$

If $\alpha_s A + h\beta_s B$ is nonsingular, (2.3) is solvable. This can be ensured by requiring $\alpha_s\beta_s \geqslant 0$, $\beta_s \neq 0$, if the system (2.1) is assumed to satisfy the asymptotic stability condition of Theorem 2.2.

We begin our study of the stability of (2.3) by considering its characteristic polynomial

$$p(z) = \det\left[\sum_{j=0}^{s} (\alpha_j A + h\beta_j B) z^j\right]. \tag{2.4}$$

We want to know under what conditions (2.4) will have no root z with $|z| \geqslant 1$, which will give the asymptotic stability of the numerical solution.

The matrices A and B can be transformed to triangular matrices simultaneously,

$$PAQ = \begin{pmatrix} a_1 & \cdots & * \\ & \ddots & * \\ & & a_n \end{pmatrix}, \qquad PBQ = \begin{pmatrix} b_1 & \cdots & * \\ & \ddots & * \\ & & b_n \end{pmatrix}, \qquad b_i \neq 0 \quad (i = 1, \dots, n).$$

Then the characteristic polynomial can be rewritten as

$$p(z) = \det P^{-1} \cdot \det \left[\sum_{j=0}^{s} \begin{pmatrix} \alpha_j a_1 + h\beta_j b_1 & \cdots & * \\ & \ddots & * \\ & & \alpha_j a_n + h\beta_j b_n \end{pmatrix} z^j \right] \det Q^{-1}. \tag{2.5}$$

Since P and Q are nonsingular, $p(z) = 0$ is equivalent to

$$\prod_{i=1}^{n} \left(\sum_{j=0}^{s} (\alpha_j a_i + h\beta_j b_i) z^j \right) = 0. \tag{2.6}$$

Thus our problem is reduced to considering the roots of (2.6).

According to the value of a_i, we have two cases. First consider the case $a_i \neq 0$ for some i. Then, for the ith term in (2.6),

$$\sum_{j=0}^{s} (\alpha_j a_i + h\beta_j b_i) z^j = a_i \sum_{j=0}^{s} \left(\alpha_j + h\beta_j \frac{b_i}{a_i} \right) z^j. \tag{2.7}$$

The characteristic polynomial of the method (2.2) applied to the standard test problem $y' = \lambda y$ is given by

$$q(z) = \sum_{j=0}^{s} (\alpha_j - h\lambda\beta_j) z^j. \tag{2.8}$$

If $h\lambda \in S_{\mathrm{R}}$, where S_{R} is the stability region of method (2.2), (2.8) will have no root on or outside the unit circle. In our problem, $\sigma = -b_i/a_i$ is just the singular value of the matrix pencil (A, B) which is in the left half-plane. If $h\sigma \in S_{\mathrm{R}}$, (2.7) will not have a root lying on or outside the unit circle.

Next consider the case when $a_i = 0$ for some i. In this case, $b_i \neq 0$ and the corresponding term in (2.6) becomes

$$\sum_{j=0}^{s} h\beta_j b_i z^j. \tag{2.9}$$

For (2.9) to have only roots inside the unit circle, we need $\sum_{j=0}^{s} \beta_j z^j$ to be a Schur polynomial. Thus we have shown

Theorem 2.4. *The multistep method* (2.2) *is asymptotically stable for asymptotically stable DAEs* (2.1) *if* $\sum_{j=0}^{s} \beta_j z^j$ *is a Schur polynomial and* $h\sigma \in S_{\mathrm{R}}$ *where* σ *are the singular values of the matrix pencil* (A, B) *and* S_{R} *is the stability region of* (2.2).

2.3. Stability of Runge–Kutta methods

The RK method for (2.1) can be written as

$$\begin{aligned} &AK_{m,i} + hB\left(x_m + \sum_{j=1}^{s} \widehat{a}_{ij} K_{m,j}\right) = 0, \quad i = 1, \ldots, s, \\ &x_{m+1} = x_m + \sum_{i=1}^{s} \widehat{b}_i K_{m,i}, \end{aligned} \tag{2.10}$$

where $K_{m,i} = [K^1_{m,i}, \ldots, K^n_{m,i}]^{\mathrm{T}}$, $i = 1, \ldots, s$, are the stage derivatives multiplied by h.

By a rearrangement of the elements of the stage derivatives

$$K_m = \left[K^1_{m,1}, \ldots, K^1_{m,s}, K^2_{m,1}, \ldots, K^2_{m,s}, \ldots, K^n_{m,1}, \ldots, K^n_{m,s}\right]^{\mathrm{T}}$$

we can rewrite (2.10)

$$\begin{pmatrix} A \otimes I_s + hB \otimes \mathcal{A} & 0 \\ -I_n \otimes \widehat{b}^{\mathrm{T}} & I_n \end{pmatrix} \begin{pmatrix} K_m \\ x_{m+1} \end{pmatrix} + \begin{pmatrix} 0 & hB \otimes e \\ 0 & -I_n \end{pmatrix} \begin{pmatrix} K_{m-1} \\ x_m \end{pmatrix} = 0, \tag{2.11}$$

where $\mathcal{A} = (\widehat{a}_{ij})$ and $\widehat{b} = [\widehat{b}_1, \ldots, \widehat{b}_s]^{\mathrm{T}}$ define the RK method and $e = [1, 1, \ldots, 1]^{\mathrm{T}}_s$.

Consider the characteristic polynomial of the difference equation (2.11)

$$p(z) = \det\left[\begin{pmatrix} (A \otimes I_s + hB \otimes \mathcal{A})z & hB \otimes e \\ -I_n \otimes \widehat{b}^{\mathrm{T}} z & I_n z - I_n \end{pmatrix}\right] = \det\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}. \tag{2.12}$$

We need the conditions under which (2.12) will have all its roots inside the unit circle. Equivalently, we consider under what conditions $|z| \geqslant 1$ gives $p(z) \neq 0$.

Again, we use the fact that for a solvable DAE, A, B can be transformed simultaneously to lower triangular matrices by nonsingular matrices P, Q. Thus it suffices to consider systems (2.1) where A, B are triangular. Here we just consider RK methods with nonsingular $\mathcal{A}$.

Under the conditions already given, $T_{11} = (A \otimes I_s + hB \otimes \mathcal{A})z$ is nonsingular when h is small, since $\mathcal{A}$ is nonsingular and the solvability of (2.1) implies the pencil (A, B) is regular, as has been pointed out in [5]. So we have

$$\begin{aligned} p(z) &= \det\begin{bmatrix} T_{11} & 0 \\ T_{12} & I \end{bmatrix} \det\begin{bmatrix} I & T_{11}^{-1}T_{12} \\ 0 & T_{22} - T_{21}T_{11}^{-1}T_{12} \end{bmatrix} \\ &= \det[T_{11}] \det\left[T_{22} - T_{21}T_{11}^{-1}T_{12}\right] \\ &= \det[T_{11}] q(z), \end{aligned} \tag{2.13}$$

where

$$q(z) = \det\left[T_{22} - T_{21}T_{11}^{-1}T_{12}\right] = \det[I_n z - M], \tag{2.14}$$

$$M = I_n - h\left(I_n \otimes \widehat{b}^{\mathrm{T}}\right)(A \otimes I_s + hB \otimes \mathcal{A})^{-1}(B \otimes e). \tag{2.15}$$

The matrix M does not depend on z and also from (2.14), it is clear that the zeros of $q(z)$ are eigenvalues of M. So we just prove that the eigenvalues of M are inside the unit circle. Expanding (2.15) using

$$(A \otimes I_s + hB \otimes \mathcal{A})^{-1} = \begin{pmatrix} (a_1 I + hb_1\mathcal{A})^{-1} & 0 & \cdots & 0 \\ * & (a_2 I + hb_2\mathcal{A})^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & (a_n I + hb_n\mathcal{A})^{-1} \end{pmatrix}$$

(where a_i, b_i $(i = 1, \dots, n)$ are diagonal elements of A, B) and by the definition of the Kronecker tensor, we arrive at

$$M = \begin{bmatrix} 1 - \widehat{b}^{\mathrm{T}}(a_1 I + hb_1 C)^{-1} hb_1 e & \cdots & 0 \\ * & \cdots & 0 \\ \vdots & \ddots & \vdots \\ * & \cdots & 1 - \widehat{b}^{\mathrm{T}}(a_n I + hb_n C)^{-1} hb_n e \end{bmatrix}.$$

Recall that the stability function of the RK method defined by $\mathcal{A}$, $\widehat{b}^{\mathrm{T}}$ is given by

$$R(\widehat{z}) = 1 + \widehat{z}\widehat{b}^{\mathrm{T}}(I - \widehat{z}\mathcal{A})^{-1} e. \tag{2.16}$$

As in Section 2.2, there are two cases according to the value of a_i. If $a_i \neq 0$ for some i, then $\sigma_i = -b_i/a_i$ is a singular value of the matrix pencil (A, B) and as pointed out before, if the DAE is asymptotically stable, it will have nonzero negative real part. Thus in this case, if $h\sigma_i \in S_{\mathrm{R}}$, where S_{R} is the stability region of the RK method with coefficient matrices $\mathcal{A}$ and $\widehat{b}^{\mathrm{T}}$, we have

$$|R(h\sigma_i)| = \left|1 + \widehat{b}^{\mathrm{T}}\big(I - (-hb_i/a_i)\mathcal{A}\big)^{-1}(-hb_i/a_i)e\right| < 1.$$

This adds no extra requirement on our RK method for stability. The second case is that $a_i = 0$ for some i. In this case, we have

$$1 - \widehat{b}^{\mathrm{T}}(a_i I + hb_i C)^{-1} hb_i e = 1 - \widehat{b}^{\mathrm{T}} b_i^{-1}\mathcal{A}^{-1} b_i e = 1 - \widehat{b}^{\mathrm{T}}\mathcal{A}^{-1} e$$

which requires the RK method to satisfy

$$\left|1 - \widehat{b}^{\mathrm{T}}\mathcal{A}^{-1} e\right| < 1. \tag{2.17}$$

This is just the requirement for strict stability for RK methods, as pointed out in [5]. Thus we get the following theorem concerning the asymptotic stability of the RK method.

Theorem 2.5. *The RK method* (2.10) *with $\mathcal{A}$ nonsingular for the asymptotically stable system* (2.1) *is asymptotically stable if it satisfies the strict stability condition* (2.17) *and if $h\sigma_i$ is in the stability region of* (2.10) *for all i, where σ_i are the singular values of* (2.1).

3. Asymptotic stability results for DDAE

In this section, we are concerned with linear delay differential algebraic equations of the form

$$Ax' + Bx + Cx'(t-\tau) + Dx(t-\tau) = 0, \tag{3.1}$$

where τ is the delay and $A, B, C, D \in \mathbb{R}^{n\times n}$ are constant coefficient matrices, and A is singular. An initial function $\phi(t)$ must be specified on the time interval $(-\tau, 0)$ for a unique solution of (3.1). Here, since only the asymptotic stability of the null solution of (3.1) is considered, we consider $\phi(t)$ such that its value is in a neighborhood of the null solution for $t \in (-\tau, 0)$.

As has been pointed out in [2,7,8], Eq. (3.1), though in the form of a neutral or retarded equation, may in fact be of advanced type. However, as we will see later, under some conditions on the coefficient matrices, the system can only be of retarded or neutral type and thus it's enough to consider only the exponential solutions of the system for asymptotic stability.

3.1. Asymptotic stability of the solution of the DDAE

The asymptotic stability of the solution of system (3.1) can be defined similarly as in Section 2.1. We require that the initial function $\phi(t)$ lies in a neighborhood of the null solution.

As we know, the solution of a linear differential equation can be studied via the characteristic equation of the original system. In our problem, the system (3.1) can be reduced to a linear essential underlying delay ODE [1] and the stability of its exponential solutions can be studied via the characteristic equation. However, we now consider directly the stability of the exponential solutions to (3.1), assuming existence and uniqueness of a continuous solution are satisfied.

Considering exponential solutions of (3.1) of the form $c\mathrm{e}^{st}$, the characteristic equation of (3.1) can be easily obtained

$$P = \det\big[(sA + B) + (sC + D)\exp(-\tau s)\big] = 0. \tag{3.2}$$

To prove the asymptotic stability of the solution, we need only show that the roots s of (3.2) all have negative real part bounded away from 0. We first give a well-known lemma that can be found in many papers [13,17].

Lemma 3.1. *Given a polynomial $P(s,z)$ in the complex variables s and z such that the following conditions are satisfied:*

(i) $P(s,0) \neq 0$ *for s such that* $\operatorname{Re} s \geqslant 0$,

(ii) $P(s,z) \neq 0$ *for (s,z) such that* $\operatorname{Re} s = 0$ *and* $|z| \leqslant 1$,

then we have $P(s,z) \neq 0$ for (s,z) such that $\operatorname{Re} s \geqslant 0$ *and* $|z| \leqslant 1$.

Let $P(s,z) = \det[(sA + B) + (sC + D)z]$. We have $P(s,0) = \det[sA + B]$. Condition (i) in Lemma 3.1 implies that the matrix pencil (A, B) cannot have a singular value with real part greater than or equal to 0. Thus the matrix $sA + B$ is nonsingular for $\operatorname{Re} s \geqslant 0$, and $P(s,z)$ can be written

$$P(s,z) = \det\big[(sA + B)\big]\det\big[I + (sA + B)^{-1}(sC + D)z\big], \quad \text{for } \operatorname{Re} s \geqslant 0.$$

If we require that

$$\sup_{\operatorname{Re} s \geqslant 0} \rho\big[(sA + B)^{-1}(sC + D)\big] < 1,$$

where ρ denotes the spectral radius, the second condition of Lemma 3.1 will also be satisfied since $\operatorname{Re} s \geqslant 0$ implies $|z| = |e^{-s\tau}| \leqslant 1$ automatically. Thus the result of Lemma 3.1 holds and the roots of the characteristic equation (3.1) will all have negative real part. This also ensures the system (3.1) can only be of retarded or neutral type since the characteristic equation of an advanced type system has an infinite number of roots with arbitrarily large real part.

To prove that the real parts of the roots of $P(s,z)$ are bounded away from 0, i.e., there is a small number $\delta > 0$ such that $P(s,z) \neq 0$ when $\operatorname{Re} s > -\delta$, we need some additional assumptions. Using the similar arguments of [4] and [20], it can be shown that when the following condition holds

$$|u^{\mathrm{T}} A u| \geqslant |u^{\mathrm{T}} C u| \quad \text{for all } u \in \mathbb{R}^n,$$

the roots of the characteristic equation (3.1) are bounded away from the imaginary axis. However, this condition is not used in proving the asymptotic stability of the numerical solution.

Theorem 3.2. *If the coefficient matrices of* (3.1) *satisfy the following conditions:*

$$\text{the matrix pencil } (A,B) \text{ only has singular values with negative real part,} \tag{3.3.1}$$

$$\sup_{\operatorname{Re} s \geqslant 0} \rho\big[(sA+B)^{-1}(sC+D)\big] < 1, \tag{3.3.2}$$

$$|u^{\mathrm{T}} A u| \geqslant |u^{\mathrm{T}} C u| \quad \text{for all } u \in \mathbb{R}^n, \tag{3.3.3}$$

then system (3.1) *is asymptotically stable.*

Corollary 3.3. *System* (3.1) *is asymptotically stable under the conditions that*

$$\text{the matrix pencil } (A,B) \text{ only has singular values with negative real part,} \tag{3.3.1$'$}$$

$$\sup_{\operatorname{Re} s = 0} \rho\big[(sA+B)^{-1}(sC+D)\big] < 1, \tag{3.3.2$'$}$$

$$|\langle u, Au\rangle| \geqslant |\langle u, Cu\rangle| \quad \text{for all } u \in \mathbb{R}^n. \tag{3.3.3$'$}$$

The corollary is a direct result of the maximum principle in complex analysis. For a proof of the inequality

$$\sup_{\operatorname{Re} s \geqslant 0} \rho\big[(sA+B)^{-1}(sC+D)\big] \leqslant \sup_{\operatorname{Re} s = 0} \rho\big[(sA+B)^{-1}(sC+D)\big],$$

refer to [14], for example.

3.2. Stability of θ-methods

We first give a definition of asymptotic stability for numerical methods.

Definition 3.1. The solution x_n of a numerical method for an asymptotically stable system (3.1) is also asymptotically stable iff for some constant b, such that if $|x_0| < b$, then $x_n \to 0$ when $n \to \infty$.

For the linear systems considered here, the constant b can be any fixed positive value and is not used in the proofs.

The θ-method applied to (3.1) gives

$$A\frac{x_{n+1}-x_n}{h}+\theta Bx_{n+1}+C\frac{x^h((n+1)h-\tau)-x^h(nh-\tau)}{h}+\theta Dx^h\big((n+1)h-\tau\big)$$
$$+(1-\theta)Bx_n+(1-\theta)Dx^h(nh-\tau)=0, \tag{3.4}$$

where $x^h(t)$ with $t>0$ is defined by piecewise linear interpolation

$$x^h(t)=\frac{t-kh}{h}x_{k+1}+\frac{(k+1)h-t}{h}x_k, \tag{3.5}$$

for $kh<t\leqslant(k+1)h,\ k=0,1,\dots$. In fact (3.5) gives

$$\begin{aligned} x^h\big((n+1)h-\tau\big)&=\delta x_{n+2-m}+(1-\delta)x_{n+1-m},\\ x^h(nh-\tau)&=\delta x_{n+1-m}+(1-\delta)x_{n-m}, \end{aligned} \tag{3.6}$$

where $0\leqslant\delta=m-\tau h^{-1}<1$ and m is the smallest integer with $\tau h^{-1}\leqslant m$.

Expanding (3.4) using (3.6), we eventually arrive at

$$\begin{aligned} (A+\theta hB)x_{n+1}=&\big(A+(\theta-1)hB\big)x_n-\delta(\theta hD+C)x_{n+2-m}\\ &-\big[(\theta hD+C)(1-\delta)+\big((1-\theta)hD-C\big)\delta\big]x_{n+1-m}\\ &-\big[(1-\theta)hD-C\big](1-\delta)x_{n-m}. \end{aligned} \tag{3.7}$$

For (3.7) to be solvable, we require that $A+\theta hB$ nonsingular for $\theta h>0$, but this is just the condition (3.3.1) of Theorem 3.2. To study the stability of the difference equation (3.7), we first obtain its characteristic equation

$$\begin{aligned} \det\big[&\big((A+\theta hB)z^{m+1}-\big(A-(1-\theta)hB\big)\big)z^m+\delta(\theta hD+C)z^2\\ &+\big((\theta hD+C)(1-\delta)+\big((1-\theta)hD-C\big)\delta\big)z+\big((1-\theta)hD-C\big)(1-\delta)\big]=0. \end{aligned} \tag{3.8}$$

Define the polynomials $P(z)$ and $Q(z,\delta)$ by

$$\begin{aligned} P(z)&=(A+\theta hB)z-\big(A-(1-\theta)hB\big),\\ Q(z,\delta)&=\big(\delta z+(1-\delta)\big)\big((z-1)C+\big(\theta z+(1-\theta)\big)hD\big). \end{aligned} \tag{3.9}$$

Then the characteristic equation (3.8) can be written as

$$\det\big[z^mP(z)+Q(z,\delta)\big]=0. \tag{3.10}$$

We give the main result of this section here.

Theorem 3.4. *For systems* (3.1) *satisfying the conditions of Theorem* 3.2, *the θ-method is asymptotically stable if $\theta\in(1/2,1]$.*

We just need to prove that under the conditions of Theorem 3.2, the roots of the characteristic equation (3.10) have modulus less than 1. We first prove the following lemma.

Lemma 3.5. *$P(z)$ is nonsingular iff $\theta\in(1/2,1]$, $|z|\geqslant1$ and condition* (3.3.1) *holds.*

Proof. Let $\lambda(A,B)$ denote the singular values of the matrix pencil (A,B). From condition (3.3.1) of Theorem 3.2, we have $\operatorname{Re}\lambda(A,B)<0$. We consider two cases. First consider the case $z\neq 1$, $|z|\geqslant 1$. Then rewrite $P(z)$ as

$$P(z)=\big[(z-1)I\big]\left[A+\left(\frac{\theta zh+(1-\theta)h}{z-1}\right)B\right].$$

Since $z\neq 1$, $\det[(z-1)I]\neq 0$. A direct calculation shows that

$$\operatorname{Re}\left(\frac{\theta zh+(1-\theta)h}{z-1}\right)>0\quad\text{iff}\quad\theta>1/2,$$

which is equivalent to

$$\operatorname{Re}\left(\frac{z-1}{\theta zh+(1-\theta)h}\right)>0\quad\text{iff}\quad\theta>1/2\quad\text{and}\quad\det\left[\frac{z-1}{\theta zh+(1-\theta)h}A+B\right]\neq 0$$

from (3.3.1). So eventually we have

$$\det\left[A+\left(\frac{\theta zh+(1-\theta)h}{z-1}\right)B\right]\neq 0$$

and hence $\det P(z)\neq 0$. The second case is when $z=1$. In this case we have $P(z)=hB$. When $h\neq 0$, $P(z)$ is nonsingular since B is a nonsingular matrix, i.e., 0 is not a singular value of (A,B). $\square$

Proof of Theorem 3.4. To prove that method (3.4) is asymptotically stable, we just need to prove that $\det[z^mP(z)+Q(z,\delta)]$ is a Schur polynomial. A sufficient condition is that $P(z)$ is invertible and $\rho[P^{-1}(z)Q(z,\delta)]<1$ for $|z|=1$, according to the main result of [17].

First we note that

$$\begin{aligned}\frac{P(z)}{\theta z+1-\theta}&=\frac{z-1}{\theta z+1-\theta}A+hB,\\ \frac{Q(z,\delta)}{\theta z+1-\theta}&=(\delta z+1-\delta)\left(hD+\frac{z-1}{\theta z+1-\theta}C\right),\end{aligned}\tag{3.11}$$

since when $\theta>1/2$ and $|z|\geqslant 1$, we have $\theta z+1-\theta\neq 0$.

Because $P(z)$ is nonsingular when $\theta>1/2$ and $|z|\geqslant 1$, we have

$$\rho\big[P^{-1}(z)Q(z,\delta)\big]=|\delta z+1-\delta|\rho\left[\left(\frac{z-1}{\theta z+1-\theta}A+hB\right)^{-1}\left(\frac{z-1}{\theta z+1-\theta}C+hD\right)\right],$$

for $|z|=1$.

However, for $0\leqslant\delta<1$ and $|z|=1$, we have $|\delta z+1-\delta|\leqslant 1-\delta+\delta|z|=1$ and thus

$$\rho\big[P^{-1}(z)Q(z,\delta)\big]\leqslant\rho\left[\left(\frac{z-1}{\theta z+1-\theta}A+hB\right)^{-1}\left(\frac{z-1}{\theta z+1-\theta}C+hD\right)\right].\tag{3.12}$$

Letting

$$\xi=\frac{z-1}{\theta z+1-\theta},\qquad\gamma=\frac{1-\theta}{\theta},$$

we have that when $\theta \in (1/2, 1]$, γ satisfies $0 \leqslant \gamma < 1$ and

$$\operatorname{Re}\xi = \operatorname{Re}\frac{1}{\theta}\frac{z-1}{z+\gamma} = \frac{1}{\theta}\operatorname{Re}\frac{z-1}{z+\gamma} \geqslant 0 \quad \text{for } |z| = 1 \text{ and } 0 \leqslant \delta < 1.$$

Thus we know

$$\xi = \frac{z-1}{\theta z + 1 - \theta}$$

transforms the unit circle $\{z\colon |z| \leqslant 1\}$ into $\{\xi\colon \operatorname{Re}\xi \geqslant 0\}$. So (3.12) is equivalent to

$$\rho\big[P^{-1}(z)Q(z,\delta)\big] \leqslant \rho\big[(\xi A + hB)^{-1}(\xi C + hD)\big] \quad \text{for } \operatorname{Re}\xi \geqslant 0. \tag{3.13}$$

However from condition (3.3.2′) of Corollary 3.3, we have

$$\sup_{\operatorname{Re}\xi \geqslant 0} \rho\big[(\xi A + hB)^{-1}(\xi C + hD)\big] < 1.$$

Hence, from (3.13),

$$\rho\big[P^{-1}(z)Q(z,\delta)\big] < 1 \quad \text{for } |z| = 1, \tag{3.14}$$

which completes the proof of Theorem 3.4. □

3.3. Stability of BDF methods

Consider an s-step BDF method applied to (3.1). As in [5], we get

$$A\frac{\rho x_n}{h} + Bx_n + C\frac{\rho x_{n-m}}{h} + Dx_{n-m} = 0, \tag{3.15}$$

where

$$\frac{\rho x_n}{h} = \frac{1}{h}\sum_{j=0}^{s} \alpha_j x_{n-j}$$

and now we assume $m = \tau/h$ for simplicity. Then the characteristic polynomial of (3.15) is given by

$$\begin{aligned} P(z) &= \det\left[\sum_{j=0}^{s} \alpha_j A z^{-j} + hBz^0 + \sum_{j=0}^{s} \alpha_j C z^{-m-j} + hDz^{-m}\right] \\ &= \det\left[\left(\sum_{j=0}^{s} \alpha_j z^{-j}\right)A + hB + z^{-m}\left[\left(\sum_{j=0}^{s} \alpha_j z^{-j}\right)C + hD\right]\right]. \end{aligned} \tag{3.16}$$

Theorem 3.6. *An A-stable BDF method is asymptotically stable for systems* (3.1) *satisfying the conditions of Theorem* 3.2.

Proof. To prove that scheme (3.15) is asymptotically stable when the corresponding BDF method is A-stable, we just need to prove under this assumption that the characteristic polynomial (3.16) has no root z with $|z| \geqslant 1$, i.e., $P(z) \neq 0$ when $|z| \geqslant 1$.

We first prove that

$$\operatorname{Re}\left(\sum_{j=0}^{s}\alpha_j z^{-j}\right) \geqslant 0 \quad \text{for } |z| \geqslant 1$$

when the corresponding BDF method is A-stable. Consider the BDF method applied to the test equation $y' = \lambda y$ with $\operatorname{Re}\lambda \leqslant 0$. The characteristic polynomial is given by

$$p(z) = \sum_{j=0}^{s}\alpha_j z^{-j} - h\lambda.$$

Since the BDF method is A-stable, for every $\xi = h\lambda$ with $\operatorname{Re}\xi \leqslant 0$, we have $p(z) \neq 0$ for $|z| \geqslant 1$. We can conclude from this fact that

$$\operatorname{Re}\left(\sum_{j=0}^{s}\alpha_j z^{-j}\right) \geqslant 0.$$

We prove this by contradiction. Suppose that there exists a z with $|z| \geqslant 1$ such that

$$\sum_{j=0}^{s}\alpha_j z^{-j} = a + ib \quad \text{where } a < 0.$$

Then from the A-stability of the method, we always can find $\xi = h\lambda = a + ib,\ a < 0$, in the stability region of the method but still satisfying

$$p(z) = \sum_{j=0}^{s}\alpha_j z^{-j} - \xi = 0 \quad \text{with } |z| \geqslant 1.$$

So

$$\operatorname{Re}\left(\sum_{j=0}^{s}\alpha_j z^{-j}\right) \geqslant 0 \quad \text{for } |z| \geqslant 1$$

and thus from condition (3.3.1′) of Corollary 3.3,

$$\left(\sum_{j=0}^{s}\alpha_j z^{-j}\right)A + hB$$

is nonsingular and

$$P(z) = \det\left[\left(\sum_{j=0}^{s}\alpha_j z^{-j}\right)A + hB\right]$$
$$\cdot \det\left[I + z^{-m}\left(\left(\sum_{j=0}^{s}\alpha_j z^{-j}\right)A + hB\right)^{-1}\left(\left(\sum_{j=0}^{s}\alpha_j z^{-j}\right)C + hD\right)\right].$$

However, from condition (3.3.2′) of Corollary 3.3, we also have

$$\det\left[I + z^{-m}\left(\left(\sum_{j=0}^{s}\alpha_j z^{-j}\right)A + hB\right)^{-1}\left(\left(\sum_{j=0}^{s}\alpha_j z^{-j}\right)C + hD\right)\right] \neq 0$$

since $|z^{-m}| \leqslant 1$ when $|z| \geqslant 1$. □

3.4. General multistep methods

In this section, we consider the stability of general multistep methods for the linear system (3.1). However, we have to make an additional assumption. As we know [5], for a linear constant coefficient DAE system to be solvable, it must be regular and thus can be represented in canonical form. As we have shown in Section 2, this is equivalent to requiring that A, B can be transformed simultaneously to triangular matrices by nonsingular constant matrices P and Q. We assume here that the matrices C and D can also be transformed to triangular form by these two matrices. This assumption is true for Hessenberg DDAEs.

Hence in the following discussion, we will only consider the case that all the coefficient matrices are upper triangular. We use the notation of Section 2 to denote these triangular matrices.

Considering again the characteristic polynomial of (3.1), we have, under the assumption of this section

$$P(s,z) = \det\big[(sA + B) + (sC + D)z\big] = \prod_{i=1}^{d}\big[s(a_i + c_i z) + (b_i + d_i z)\big], \tag{3.17}$$

where a_i, b_i, c_i, d_i $(i = 1,\dots,d)$ are the diagonal elements of the corresponding matrices and d is the dimension of the problem and $z = \exp(-\tau s)$. We have the following proposition to ensure the asymptotic stability of (3.1).

Proposition 3.7. *System* (3.1) *is asymptotically stable if*
- (i) *for any i, if $a_i = 0$, then $c_i = 0$.*
- (ii) *$|a_i| > |c_i|$ for all i such that $a_i \neq 0$, $|b_i| > |d_i|$ for all i such that $a_i = 0$ and when $a_i \neq 0$,* $\mathrm{Re}\,((a_i + c_i z)^{-1}(b_i + d_i z)) > 0$ *for $|z| \leqslant 1$.*
- (iii) $|u^{\mathrm{T}}Au| \geqslant |u^{\mathrm{T}}Cu|$ *for all $u \in \mathbb{R}^n$.*

Proof. Notice that when $\mathrm{Re}\, s \geqslant 0$, $|z| \leqslant 1$. We need to prove that under the conditions above, $P(s,z) \neq 0$ when $\mathrm{Re}\, s \geqslant 0$. First consider the case that $a_i = c_i = 0$ for some i. In this case, we have $b_i + d_i z \neq 0$ from $|b_i| > |d_i|$ and $|z| \leqslant 1$ ($\mathrm{Re}\, s \geqslant 0$). In the case $a_i \neq 0$, we have $a_i + c_i z \neq 0$ from $|a_i| > |c_i|$ and thus

$$s(a_i + c_i z) + (b_i + d_i z) = (a_i + c_i z)\left(s + \frac{b_i + d_i z}{a_i + c_i z}\right) \neq 0$$

since Re $(a_i + c_i z)^{-1}(b_i + d_i z) > 0$ for $|z| \leqslant 1$. □

Condition (i) of Proposition 3.7 is also necessary for the system (3.1) to be retarded or neutral.

Consider a general multistep method

$$\sum_{j=0}^{s} \alpha x_{n+j} = h \sum_{j=0}^{s} \beta_j f_{n+j}. \tag{3.18}$$

When (3.18) is applied to (3.1), we have

$$\sum_{j=0}^{s} (\alpha_j A x_{n+j} + h\beta_j B x_{n+j} + \alpha_j C x_{n+j-m} + h\beta_j D x_{n+j-m}) = 0, \tag{3.19}$$

where m has the same meaning as before.

Again, we consider the characteristic polynomial of (3.19), using the fact that all the coefficient matrices are triangular. We prove that the polynomial will be nonzero when $|z| \geqslant 1$.

$$\begin{aligned} p(z) &= \det \left[\sum_{j=0}^{s} \left(\alpha_j A z^j + h\beta_j B z^j + \alpha_j C z^{j-m} + h\beta_j D z^{j-m} \right) \right] \\ &= \prod_{i=1}^{d} \sum_{j=0}^{s} \left((a_i + c_i z^{-m}) \alpha_j + h(b_i + d_i z^{-m}) \beta_j \right) z^j. \end{aligned} \tag{3.20}$$

Consider an arbitrary term in (3.20). If $a_i \neq 0$ then we rewrite it as

$$\sum_{j=0}^{s} \left((a_i + c_i z^{-m}) \alpha_j + h(b_i + d_i z^{-m}) \beta_i \right) z^j = (a_i + c_i z^{-m}) \sum_{j=0}^{s} \left(\alpha_j + h \frac{b_i + d_i z^{-m}}{a_i + c_i z^{-m}} \beta_i \right) z^j,$$

since when $|z| \geqslant 1$ we have $a_i + c_i z^{-m} \neq 0$ from the condition of Proposition 3.7. Thus because we already have

$$\mathrm{Re}\left(-\frac{b_i + d_i z^{-m}}{a_i + c_i z^{-m}} \right) \leqslant 0$$

then if

$$-h \frac{b_i + d_i z^{-m}}{a_i + c_i z^{-m}} \in S_{\mathrm{R}}$$

where S_{R} is the stability region of the corresponding multistep method,

$$\sum_{j=0}^{s} \left((a_i + c_i z^{-m}) \alpha_j + h(b_i + d_i z^{-m}) \beta_j \right) z^j \neq 0.$$

In the case $a_i = 0$ for some i, the corresponding term of (3.20) reduces to

$$\left(\sum_{j=0}^{s} \beta_j z^j \right) (b_i + d_i z^{-m}).$$

If $\sum_{j=0}^{s}\beta_j z^j$ is a Schur polynomial, this term is nonzero from the condition of Proposition 3.7. This finishes the proof of the following theorem.

Theorem 3.8. *If the DDAE system* (3.1) *satisfies the conditions of Proposition* 3.7 *and also the condition that*

$$-h\frac{b_i + d_i z}{a_i + c_i z} \in S_{\mathrm{R}} \quad \text{for } |z| \geqslant 1,$$

then if the multistep method satisfies that $\sum_{j=0}^{s}\beta_j z^j$ *is a Schur polynomial, the solution of the multistep method is asymptotically stable.*

3.5. Asymptotic stability of Runge–Kutta methods

Consider now the RK method for (3.1). We have

$$\begin{aligned} &AK_{n,i} + hB\left(x_n + \sum_{j=1}^{s}\widehat{a}_{ij}K_{n,j}\right) + CK_{n-m,j} + hD\left(x_{n-m} + \sum_{j=1}^{s}\widehat{a}_{ij}K_{n-m,j}\right) = 0 \\ &\quad \text{for } i = 1,\dots,s, \\ &x_{n+1} = x_n + \sum_{i=1}^{s}\widehat{b}_i K_{n,i}, \end{aligned} \tag{3.21}$$

where $K_{n,i} = [K_{n,i}^1,\dots,K_{n,i}^d]^{\mathrm{T}}$, $i = 1,\dots,s$, are stage derivatives multiplied by h. Denoting again

$$\widehat{b}^{\mathrm{T}} = \left[\widehat{b}_1, \widehat{b}_2, \dots, \widehat{b}_s\right], \qquad \mathcal{A} = (\widehat{a}_{ij}),$$

we assume that the eigenvalues of $\mathcal{A}$ all have positive real part. Such RK methods do exist. A simple example is the semi-explicit RK method with positive diagonal coefficients. Another example is the implicit two-stage fourth order Butcher–Kuntzmann formula, [13]. After a rearrangement of the variables of the stage derivatives as

$$K_n = \left[K_{n,1}^1,\dots,K_{n,s}^1, K_{n,1}^2,\dots,K_{n,s}^2,\dots,K_{n,1}^d,\dots,K_{n,s}^d\right]^{\mathrm{T}},$$

we can rewrite (3.21) in the form

$$\begin{aligned} &\begin{pmatrix} A\otimes I_s + hB\otimes\mathcal{A} & 0 \\ -I_d\otimes\widehat{b}^{\mathrm{T}} & I_d \end{pmatrix}\begin{pmatrix} K_n \\ x_{n+1}\end{pmatrix} + \begin{pmatrix} 0 & hB\otimes e \\ 0 & -I_d\end{pmatrix}\begin{pmatrix} K_{n-1} \\ x_n\end{pmatrix} \\ &\quad + \begin{pmatrix} C\otimes I_s + hD\otimes\mathcal{A} & 0 \\ 0 & 0\end{pmatrix}\begin{pmatrix} K_{n-m} \\ x_{n+1-m}\end{pmatrix} + \begin{pmatrix} 0 & hD\otimes e \\ 0 & 0\end{pmatrix}\begin{pmatrix} K_{n-1-m} \\ x_{n-m}\end{pmatrix} = 0, \end{aligned} \tag{3.22}$$

where $e = [1,1,\dots,1]_s^{\mathrm{T}}$.

The characteristic polynomial of (3.22) is given by

$$p(z) = \det\left[z^{m+1}\begin{pmatrix} A\otimes I_s + hB\otimes\mathcal{A} & 0 \\ -I_d\otimes\widehat{b}^{\mathrm{T}} & I_d\end{pmatrix} + z^m\begin{pmatrix} 0 & hB\otimes e \\ 0 & -I_d\end{pmatrix}\right.$$

$$+\ z\begin{pmatrix} C\otimes I_s + hD\otimes\mathcal{A} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & hD\otimes e \\ 0 & 0 \end{pmatrix}\Bigg] = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, \tag{3.23}$$

where

$$\begin{aligned} T_{11} &= z^{m+1}(A\otimes I_s + hB\otimes\mathcal{A}) + z(C\otimes I_s + hD\otimes\mathcal{A}), \\ T_{12} &= z^m hB\otimes e + hD\otimes e, \\ T_{21} &= -z^{m+1} I_d\otimes\widehat{b}^{\mathrm{T}}, \\ T_{22} &= z^{m+1} I_d - z^m I_d. \end{aligned} \tag{3.24}$$

We first prove that, under the conditions of Proposition 3.7, T_{11} is nonsingular. Using the fact that all the coefficient matrices are of upper triangular form, we have

$$\begin{aligned} \det[T_{11}] &= \prod_{i=1}^{d} \det\big[z^{m+1}(a_i I + hb_i\mathcal{A}) + z(c_i I + hd_i\mathcal{A})\big] \\ &= \prod_{i=1}^{d} \det\big[z^{m+1}(a_i + c_i z^{-m})I + z^{m+1}(b_i + d_i z^{-m})\mathcal{A}\big]. \end{aligned} \tag{3.25}$$

For each term in $\det[T_{11}]$, if $a_i \neq 0$ it can be rewritten as

$$q(z) = \det\left[z^{m+1}(a_i + c_i z^{-m})\left(I + h\frac{b_i + d_i z^{-m}}{a_i + c_i z^{-m}}\mathcal{A}\right)\right]. \tag{3.26}$$

When $|z| \geqslant 1$, we have

$$a_i + c_i z^{-m} \neq 0, \qquad \operatorname{Re}\frac{b_i + d_i z^{-m}}{a_i + c_i z^{-m}} > 0$$

from the conditions of Proposition 3.7. Thus $q(z) \neq 0$ from the fact that all the eigenvalues of $\mathcal{A}$ have positive real part. For each term with $a_i = c_i = 0$, we have

$$q(z) = \det\big[z^{m+1}h(b_i + d_i z^{-m})\mathcal{A}\big] \neq 0$$

since $b_i + d_i z^{-m} \neq 0$ for $|z| \geqslant 1$. Thus we have shown that T_{11} is nonsingular. Hence $p(z)$ can be rewritten as

$$p(z) = \det[T_{11}]\det\big[T_{22} - T_{21}T_{11}^{-1}T_{12}\big]$$

where T_{11}^{-1} is upper triangular with diagonal elements

$$z^{-(m+1)}\big((a_i + c_i z^{-m})I + h(b_i + d_i z^{-m})\mathcal{A}\big)^{-1}, \quad i = 1, 2, \ldots, d.$$

By direct calculation, we have

$$\begin{aligned} &\det\big[T_{22} - T_{21}T_{11}^{-1}T_{12}\big] \\ &\quad = \prod_{i=1}^{d} z^m\big[z - \big(1 - \widehat{b}^{\mathrm{T}}\big((a_i + c_i z^{-m})I + h(b_i + d_i z^{-m})\mathcal{A}\big)^{-1}h(b_i + d_i z^{-m})e\big)\big]. \end{aligned} \tag{3.27}$$

Consider the terms in (3.27). When $a_i \neq 0$ for some i, we have

$$1 - \widehat{b}^{\mathrm{T}}\big((a_i + c_i z^{-m})I + h(b_i + d_i z^{-m})\mathcal{A}\big)^{-1} h(b_i + d_i z^{-m})e$$
$$= 1 - \widehat{b}^{\mathrm{T}}\left(I + h\frac{b_i + d_i z^{-m}}{a_i + c_i z^{-m}}\mathcal{A}\right)^{-1} h\frac{b_i + d_i z^{-m}}{a_i + c_i z^{-m}}e = R\left(-h\frac{b_i + d_i z^{-m}}{a_i + c_i z^{-m}}\right),$$

where

$$R(\widehat{z}) = 1 - \widehat{z}\widehat{b}^{\mathrm{T}}(I - \widehat{z}\mathcal{A})^{-1}e$$

is the stability function of the RK method. Since

$$\mathrm{Re}\left(-\frac{b_i + d_i z^{-m}}{a_i + c_i z^{-m}}\right) \leqslant 0,$$

if we also have

$$-h\frac{b_i + d_i z^{-m}}{a_i + c_i z^{-m}} \in S_{\mathrm{R}} \quad \text{for } |z| \geqslant 1,$$

where S_{R} is the stability region of the RK method, then we have

$$1 - \widehat{b}^{\mathrm{T}}\big((a_i + c_i z^{-m})I + h(b_i + d_i z^{-m})\mathcal{A}\big)^{-1} h\big(b_i + d_i z^{-m}\big)e < 1.$$

Thus the terms in (3.27) corresponding to this case will be nonzero when $|z| \geqslant 1$.

In the case that $a_i = c_i = 0$ for some i, the terms in the product reduce to the form

$$1 - \widehat{b}^{\mathrm{T}}\big((a_i + c_i z^{-m})I + h(b_i + d_i z^{-m})\mathcal{A}\big)^{-1} h(b_i + d_i z^{-m})e = 1 - \widehat{b}^{\mathrm{T}}\mathcal{A}^{-1}e.$$

Thus if the RK method satisfies $|1 - \widehat{b}^{\mathrm{T}}\mathcal{A}^{-1}e| < 1$, i.e., the RK method is strictly stable, then the terms in (3.27) corresponding to this case will also be nonzero. So we proved that when $|z| \geqslant 1$, $p(z) \neq 0$, which leads immediately to the following theorem.

Theorem 3.9. *For linear systems* (3.1) *which satisfy the conditions of Proposition* 3.7 *and also the condition*

$$-h\frac{b_i + d_i z^{-m}}{a_i + c_i z^{-m}} \in S_{\mathrm{R}} \quad \textit{when } |z| \geqslant 1$$

for a strictly stable RK method for which all the eigenvalues of its coefficient matrix $\mathcal{A}$ have positive real part, which has stability region S_{R}, the numerical solution is asymptotically stable.

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