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Asymptotic stability of Hessenberg delay differential–algebraic equations of retarded or neutral type ☆

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Abstract

In this paper we prove that for Hessenberg delay DAEs of retarded type, the direct linearization along the stationary solution is valid. This validity is obtained by showing the equivalence between the direct linearization and the linearization of the state space form of the original problem, which is assured to be legitimate. Thus the study of the asymptotic stability of the stationary solution can be transformed to the study of the null solution of the linearization of the original problem. We point out here that a similar method can be used to prove the validity of the direct linearization of delay differential–algebraic equations of neutral type.

1. Introduction

Recently, the study of stability of delay ODEs (DODEs) and numerical methods for them has been an active field of research. Different kinds of stability have been defined for delay ODE systems and numerical methods as well. Stability of Runge–Kutta (RK) methods has been studied in [3,8,10,12,13,18], where scalar or systems of delay ODEs with constant or variable delays are considered. In [2,9,11,14,15], stability of θ-methods is studied for delay ODEs with different structures. Most of these results are for linear constant coefficient systems with constant delay.

Not much work has been done regarding the stability of delay differential algebraic equations (DDAEs), which have both delay and algebraic constraints. In [1,4,5,7], the structure of DDAEs and order and convergence of numerical methods have been studied but the asymptotic stability of these systems and numerical methods still remains to be investigated. In [20], we give some results on the asymptotic stability of linear DDAEs and numerical methods.

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In [17], asymptotic stability of Euler–Lagrange equations for constrained mechanical systems is studied by showing the equivalence of the direct linearization of the original system to that of its corresponding state space form. In fact their method can be extended to DAEs of Hessenberg form for index from one to three. In this paper, we will show the validity of the linearization of a nonlinear DDAE system of Hessenberg form, using the approach of [17]. Thus the asymptotic stability of a nonlinear DDAE system can be studied locally via its linearization.

It should be pointed out here that our results are valid only when the stationary solution exists and only in a neighborhood of the stationary solution. It is obvious that stationary solutions exist only when there are solutions to the nonlinear systems of the right hand sides of the DDAEs. Our study is useful in that it transforms a nonlinear problem into a linear problem which can be studied much more easily [20] and the stability of the null solution of the linear problem is closely related to the linear stability of numerical methods.

In [1], nonlinear delay differential–algebraic equations of retarded type which are extensions of Hessenberg form are given with index up to three. The index-one problem is

$$\begin{aligned} x' &= f\big(x, x(t-1), y, y(t-1)\big), \\ 0 &= g\big(x, x(t-1), y\big), \end{aligned}$$

where $\partial g/\partial y$ is nonsingular. The index-two problem is of the form

$$\begin{aligned} x' &= f\big(x, x(t-1), y\big), \\ 0 &= g(x), \end{aligned}$$

where $(\partial g/\partial x)(\partial f/\partial y)$ is nonsingular. The index-three problem is given by

$$\begin{aligned} y' &= f\big(x, x(t-1), y, y(t-1), z\big), \\ x' &= g(x, y), \\ 0 &= h(x), \end{aligned}$$

where $(\partial h/\partial x)(\partial g/\partial y)(\partial f/\partial z)$ is nonsingular. Here the delays are normalized to 1 and for conciseness we denote $x(t)$ simply as x. It is pointed out in [1] that delays are only allowed in certain variables as described above because allowing delays in other variables/equations will give rise to equations of neutral or advanced type. So for simplicity, in Section 2 we first consider DDAEs of retarded type.

Essentially, the approaches to the proofs in the following sections are the same. Since the linearization of the state space form of a DDAE is assured to be legitimate, we first obtain the formal state space form of the original DDAE by using the implicit function theorem and then we obtain the linearization of it, which is a linear delay ODE. In the second step, the direct linearization of the original DDAE is obtained and the state space form of it is obtained, which is also a linear delay ODE. Equivalence between these two linear delay ODEs is proved using the property of the stationary solutions and the implicit function theorem.

For DDAEs of neutral type, we give the result for index-one problems in Section 3. Higher index problems can be studied similarly but the approach used in this paper will be too complex for them and alternative approaches are desired.

2. Hessenberg DDAEs of retarded type

2.1. Index-one problems

Consider the index-one problem

$$\begin{aligned} x' &= f\big(x, x(t-1), y, y(t-1)\big), \\ 0 &= g\big(x, x(t-1), y\big), \end{aligned} \tag{1}$$

where $f : U_1 \to V_1$, $g : U_2 \to V_2$, $U_1 \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_y}$, $U_2 \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$, $V_1 \subseteq \mathbb{R}^{n_x}$, $V_2 \subseteq \mathbb{R}^{n_y}$ and $g_y \neq 0$. Let (x_0, y_0) be the stationary solution of (1). Then

$$f(x_0, x_0, y_0, y_0) = 0, \qquad g(x_0, x_0, y_0) = 0.$$

Consider the nonlinear constraint $g(x, x(t-1), y) = 0$. By the implicit function theorem, there exists a neighborhood Θ of (x_0, x_0) and a differentiable function $\rho(x, x(t-1))$ such that in Θ, we have $y = \rho(x, x(t-1))$ and

$$g\big(x, x(t-1), \rho\big(x, x(t-1)\big)\big) = 0$$

holds for any pair $(x, x(t-1))$ in Θ.

Thus in Θ the constraint can be satisfied naturally and the reduced state space form of (1) is given by

$$x' = f\big(x, x(t-1), \rho\big(x, x(t-1)\big), \rho\big(x(t-1), x(t-2)\big)\big). \tag{2}$$

Linearizing (2) along the stationary solution yields, with the abuse of notation x instead of ∂x,

$$\begin{aligned} x' = {}& \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial \rho}{\partial x}\right)x + \left(\frac{\partial f}{\partial x(t-1)} + \frac{\partial f}{\partial y}\frac{\partial \rho}{\partial x(t-1)} + \frac{\partial f}{\partial y(t-1)}\frac{\partial \rho}{\partial x}\right)x(t-1) \\ & + \left(\frac{\partial f}{\partial y(t-1)}\frac{\partial \rho}{\partial x(t-1)}\right)x(t-2), \end{aligned} \tag{3}$$

where the partial derivatives all take values at the stationary solution. In the rest of the paper, if not specified otherwise, all partial derivatives will be considered as taking values at the stationary solution for conciseness.

The linearization (3) is assured to give the correct local information about the stationary solution according to the theory of [6]. We now consider the direct linearization of the original problem (1) to show that in fact the two linearizations are equivalent at the stationary solution. Direct linearization of (1) at the stationary solution yields

$$x' = \frac{\partial f}{\partial x}x + \frac{\partial f}{\partial x(t-1)}x(t-1) + \frac{\partial f}{\partial y}y + \frac{\partial f}{\partial y(t-1)}y(t-1), \tag{4a}$$

$$0 = \frac{\partial g}{\partial x}x + \frac{\partial g}{\partial x(t-1)}x(t-1) + \frac{\partial g}{\partial y}y. \tag{4b}$$

Since $\partial g/\partial y \neq 0$, solve for y using (4b) and insert it back into (4a) to obtain

$$
\begin{aligned}
x' = {} & \left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\left(\frac{\partial g}{\partial y}\right)^{-1}\frac{\partial g}{\partial x}\right)x \\
& + \left(\frac{\partial f}{\partial x(t-1)} - \frac{\partial f}{\partial y}\left(\frac{\partial g}{\partial y}\right)^{-1}\frac{\partial g}{\partial x(t-1)} - \frac{\partial f}{\partial y(t-1)}\left(\frac{\partial g}{\partial y}\right)^{-1}\right)x(t-1) \\
& - \left(\frac{\partial f}{\partial y(t-1)}\left(\frac{\partial g}{\partial y}\right)^{-1}\frac{\partial g}{\partial x(t-1)}\right)x(t-2).
\end{aligned}
\tag{5}
$$

However, since $0 = g(x, x(t-1), \rho(x, x(t-1)))$ in Θ, we have

$$
\begin{aligned}
\frac{\mathrm{d}g}{\mathrm{d}x} &= \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}\frac{\partial \rho}{\partial x} = 0, \\
\frac{\mathrm{d}g}{\mathrm{d}x(t-1)} &= \frac{\partial g}{\partial x(t-1)} + \frac{\partial g}{\partial y}\frac{\partial \rho}{\partial x(t-1)} = 0,
\end{aligned}
$$

in Θ. Thus

$$
\begin{aligned}
\frac{\partial \rho}{\partial x} &= -\left(\frac{\partial g}{\partial y}\right)^{-1}\frac{\partial g}{\partial x}, \\
\frac{\partial \rho}{\partial x(t-1)} &= -\left(\frac{\partial g}{\partial y}\right)^{-1}\frac{\partial g}{\partial x(t-1)},
\end{aligned}
$$

from which we can see that linearizations (3) and (4) are the same, which means linearizing the DDAE directly along the stationary solution is valid and will give the same asymptotic stability information as that of the state space form DDE problem. However, (3) is just a formal linearization since ρ usually can not be explicitly written out, while (4) can be obtained directly from the original system.

2.2. *Hessenberg index-two problems*

We now turn to the Hessenberg index-two problem

$$x' = f\big(x, x(t-1), y\big), \tag{6a}$$

$$0 = g(x), \tag{6b}$$

where $f: U_1 \to V_1$, $g: U_2 \to V_2$, $U_1 \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$, $U_2 \subseteq \mathbb{R}^{n_x}$, $V_1 \subseteq \mathbb{R}^{n_x}$, $V_2 \subseteq \mathbb{R}^{n_y}$ and $g_x f_y \neq 0$. Let (x_0, y_0) be the stationary solution of (6). Then

$$f(x_0, x_0, y_0) = 0, \qquad g(x_0) = 0.$$

Since $g_x f_y \neq 0$, g_x has full row rank. Introducing a local coordinate chart for the constraint manifold $S = \{x\colon\ 0 = g(x)\}$ with proper ordering of the variables $x = (x_1, x_2)$, where $x_1 \in \mathbb{R}^{n_y}$, $x_2 \in \mathbb{R}^{n_x - n_y}$, $\partial g/\partial x_1 \neq 0$, there exists a neighborhood $\Theta \subset \mathbb{R}^{n_x - n_y}$ of x_{20} and a differentiable function $\rho: \mathbb{R}^{n_x - n_y} \to \mathbb{R}^{n_y}$ such that for all $x_2 \in \Theta$, we have $g(\rho(x_2), x_2) = 0$.

Note that the first derivative of (6b) yields

$$\frac{\partial g}{\partial x}x' = \frac{\partial g}{\partial x}f\big(x, x(t-1), y\big) = 0,$$

where

$$\frac{\partial g}{\partial x}\frac{\partial f}{\partial y} \neq 0.$$

Again by the implicit function theorem, y can be represented as a differentiable function of $(x, x(t-1))$ by

$$y = h(x, x(t-1))$$

in some neighborhood of (x_0, x_0) and we have

$$\frac{\partial g}{\partial x} f\big(x, x(t-1), h(x, x(t-1))\big) = 0$$

for every pair $(x, x(t-1))$ in this neighborhood. In fact, y is a function of $(x_2, x_2(t-1))$ only, since

$$h(x, x(t-1)) = h\big((\rho(x_2), x_2), (\rho(x_2(t-1)), x_2(t-1))\big).$$

So the constraints can be dropped to get the state space form DDE for free variable x_2.

To decouple the original system and rewrite it in independent coordinate variables, we let $z = (z_1, z_2)$, $z_1 \in \mathbb{R}^{n_y}$, $z_2 \in \mathbb{R}^{n_x - n_y}$ and

$$z = Z(x) = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 - \rho(x_2) \\ x_2 \end{bmatrix}.$$

We then have

$$x = X(z) = \begin{bmatrix} z_1 + \rho(z_2) \\ z_2 \end{bmatrix},$$

and

$$T = \frac{\partial x}{\partial z} = \frac{\partial X}{\partial z} = \begin{bmatrix} I_m & \dfrac{\partial \rho(z_2)}{\partial z_2} \\ 0 & I_{n-m} \end{bmatrix}$$

which is nonsingular. The same relations exist between $x(t-1)$ and $z(t-1)$.

Thus the DAE problem (6) can be transformed into the following form, which contains a DDE subproblem for variable $z_2 = x_2$,

$$T(z)z' = f\big(X(z), X(z(t-1)), h(X(z), X(z(t-1)))\big), \tag{7a}$$

$$0 = z_1, \tag{7b}$$

with stationary solution $z_0 = Z(x_0) = (0, z_{20}) = (0, x_{20})$.

In fact, since $z_1' = 0$, (7) can be written as

$$\frac{\mathrm{d}\rho(z_2)}{\mathrm{d}t} = \frac{\partial \rho(z_2)}{\partial z_2} z_2' = f_1\big(X(z), X(z(t-1)), h(X(z), X(z(t-1)))\big), \tag{8a}$$

$$z_2' = f_2\big(X(z), X(z(t-1)), h(X(z), X(z(t-1)))\big), \tag{8b}$$

$$0 = z_1, \tag{8c}$$

and we can show that (8a) will hold if (8b), (8c) hold. Since when $z_1 = 0$, we have $x_1 = \rho(x_2) = \rho(z_2)$ and hence $g(\rho(z_2), z_2) = 0$ for any z_2, this in turn gives

$$\frac{\mathrm{d}}{\mathrm{d}t} g\big(\rho(z_2), z_2\big) = \frac{\partial g}{\partial x_1} \frac{\mathrm{d}\rho(z_2)}{\mathrm{d}t} + \frac{\partial g}{\partial x_2} z_2' = 0.$$

However, we also have

$$\frac{\mathrm{d}}{\mathrm{d}t} g(x_1, x_2) = \frac{\partial g}{\partial x} f = \frac{\partial g}{\partial x_1} f_1 + \frac{\partial g}{\partial x_2} f_2 = 0.$$

So, when (8b) is satisfied, we have

$$\frac{\mathrm{d}\rho(z_2)}{\mathrm{d}t} = f_1 = -\left(\frac{\partial g}{\partial x_1}\right)^{-1} \frac{\partial g}{\partial x_2} f_2,$$

because $\partial g/\partial x_1 \neq 0$. This means (8a) is automatically satisfied when (8b), (8c) are satisfied.

Note that now the variables z_1 and z_2 are independent and linearization makes sense. To linearize (7), we first show that linearization of a system of the form

$$g(y)y' = f(y)$$

is just

$$g(y_0)y' = Df(y_0)y,$$

where y_0 is the fixed point satisfying $f(y_0) = 0$ and Df the Jacobian of f and $g(y) \neq 0,\ g \in \mathbb{R}^{n\times n}$. This is obvious since the original system is equivalent to

$$y' = g^{-1}(y)f(y).$$

For the linearization of the above equation, we find that the terms containing the derivatives of $g^{-1}(y)$ disappear since they are multiplied by $f(y)$, which is 0 at the fixed points. Multiplying both sides of the linearization of $y' = g^{-1}(y)f(y)$ by the term $g(y_0)$, we obtain the desired linearization. Even though there are delay variables on the right hand side of (7), the above discussion is still valid.

Linearizing (7) gives

$$Tz' = \left(\frac{\partial f}{\partial x} T + \frac{\partial f}{\partial y} \frac{\partial h}{\partial x} T\right) z + \left(\frac{\partial f}{\partial x(t-1)} T + \frac{\partial f}{\partial y} \frac{\partial h}{\partial x(t-1)} T\right) z(t-1), \tag{9a}$$

$$0 = z_1. \tag{9b}$$

The direct linearization of (6) is given by

$$x' = \frac{\partial f}{\partial x} x + \frac{\partial f}{\partial x(t-1)} x(t-1) + \frac{\partial f}{\partial y} y, \tag{10a}$$

$$0 = \frac{\partial g}{\partial x} x. \tag{10b}$$

Note here that the derivatives are constant. Since

$$\frac{\partial g}{\partial x} \frac{\partial f}{\partial y} \neq 0,$$

by solving for y and inserting it back into (10a), we can rewrite (10) as

$$x' = \left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\left(\frac{\partial g}{\partial x}\frac{\partial f}{\partial y}\right)^{-1}\frac{\partial g}{\partial x}\frac{\partial f}{\partial x}\right)x$$
$$+\left(\frac{\partial f}{\partial x(t-1)} - \frac{\partial f}{\partial y}\left(\frac{\partial g}{\partial x}\frac{\partial f}{\partial y}\right)^{-1}\frac{\partial g}{\partial x}\frac{\partial f}{\partial x(t-1)}\right)x(t-1), \tag{11}$$
$$0 = \frac{\partial g}{\partial x}x,$$

which in fact contains a DDE subproblem for x_2. To decouple, we use the coordinate change

$$x = T(z_0)z, \qquad x(t-1) = T(z_0)z(t-1),$$

then (11) becomes

$$Tz' = \left(\frac{\partial f}{\partial x}T - \frac{\partial f}{\partial y}\left(\frac{\partial g}{\partial x}\frac{\partial f}{\partial y}\right)^{-1}\frac{\partial g}{\partial x}\frac{\partial f}{\partial x}T\right)z$$
$$+\left(\frac{\partial f}{\partial x(t-1)}T - \frac{\partial f}{\partial y}\left(\frac{\partial g}{\partial x}\frac{\partial f}{\partial y}\right)^{-1}\frac{\partial g}{\partial x}\frac{\partial f}{\partial x(t-1)}T\right)z(t-1), \tag{12a}$$
$$0 = \frac{\partial g}{\partial x}Tz. \tag{12b}$$

We see that (12b) is equivalent to (9b) since in Θ,

$$\frac{\partial g}{\partial x_1}\frac{\partial \rho(z_2)}{\partial z_2} + \frac{\partial g}{\partial x_2} = \frac{\mathrm{d}g}{\mathrm{d}z_2} = 0, \qquad \frac{\partial g}{\partial x_1} \neq 0$$

and (12b) can be expanded as

$$0 = \frac{\partial g}{\partial x_1}z_1 + \left(\frac{\partial g}{\partial x_1}\frac{\partial \rho(z_2)}{\partial z_2} + \frac{\partial g}{\partial x_2}\right)z_2.$$

Comparing (9a) and (12a), we see that to prove that the two linearizations are the same, it is enough to prove

$$-\frac{\partial f}{\partial y}\left(\frac{\partial g}{\partial x}\frac{\partial f}{\partial y}\right)^{-1}\frac{\partial g}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}\frac{\partial h}{\partial x},$$
$$-\frac{\partial f}{\partial y}\left(\frac{\partial g}{\partial x}\frac{\partial f}{\partial y}\right)^{-1}\frac{\partial g}{\partial x}\frac{\partial f}{\partial x(t-1)} = \frac{\partial f}{\partial y}\frac{\partial h}{\partial x(t-1)}. \tag{13}$$

Note that here all the partial derivatives, including the elements of T, take values at the stationary solution. Since

$$\frac{\partial g}{\partial x}f\big(x, x(t-1), h\big(x, x(t-1)\big)\big) = 0$$

in some neighborhood of (x_0, x_0), we have, in this neighborhood,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial g}{\partial x}f\big(x, x(t-1), h\big(x, x(t-1)\big)\big)\right) = 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}x(t-1)}\left(\frac{\partial g}{\partial x}f\big(x, x(t-1), h\big(x, x(t-1)\big)\big)\right) = 0,$$

which at the stationary solution $x_0 = X(z_0)$ gives

$$\frac{\partial g}{\partial x}\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\frac{\partial f}{\partial y}\frac{\partial h}{\partial x} = 0, \qquad \frac{\partial g}{\partial x}\frac{\partial f}{\partial x(t-1)} + \frac{\partial g}{\partial x}\frac{\partial f}{\partial y}\frac{\partial h}{\partial x(t-1)} = 0,$$

and thus

$$\begin{aligned} \frac{\partial h}{\partial x} &= -\left(\frac{\partial g}{\partial x}\frac{\partial f}{\partial y}\right)^{-1}\frac{\partial g}{\partial x}\frac{\partial f}{\partial x}, \\ \frac{\partial h}{\partial x(t-1)} &= -\left(\frac{\partial g}{\partial x}\frac{\partial f}{\partial y}\right)^{-1}\frac{\partial g}{\partial x}\frac{\partial f}{\partial x(t-1)}, \end{aligned} \tag{14}$$

from which we see that (13) holds.

So for Hessenberg index-two problems, the two linearizations are equivalent and thus yield the same asymptotic stability information at the fixed points, while only the direct linearization is practical.

2.3. *Hessenberg index-three problems*

Let us now consider the more complicated index-three problem

$$y' = f\big(x, x(t-1), y, y(t-1), z\big), \tag{15a}$$

$$x' = g(x, y), \tag{15b}$$

$$0 = h(x), \tag{15c}$$

where $f: U_1 \to V_1$, $g: U_2 \to V_2$, $h: U_3 \to V_3$, $U_1 \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_z}$, $U_2 \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$, $U_3 \subseteq \mathbb{R}^{n_x}$, $V_1 \subseteq \mathbb{R}^{n_y}$, $V_2 \subseteq \mathbb{R}^{n_x}$, $V_3 \subseteq \mathbb{R}^{n_z}$ and $h_x g_y f_z \neq 0$. Let (x_0, y_0, z_0) be the stationary solution of (15). Then

$$f(x_0, x_0, y_0, y_0, z_0) = 0, \qquad g(x_0, y_0) = 0, \qquad h(x_0) = 0.$$

Since $\partial h/\partial x$ has full row rank, using the same argument as before, we have, in a neighborhood of x_{20}, a differentiable function $x_1 = \rho(x_2)$ such that $0 = h(\rho(x_2), x_2)$ holds for every x_2 in this neighborhood, where $x = (x_1, x_2)$, $x_1 \in \mathbb{R}^{n_z}$, $x_2 \in \mathbb{R}^{n_x - n_z}$. The condition $\partial h/\partial x_1 \neq 0$ defines a partition of x.

The first derivative of the constraint (15c) gives

$$\frac{\partial h}{\partial x}x' = \frac{\partial h}{\partial x}g(x, y) = 0.$$

Now since $(\partial h/\partial x)(\partial g/\partial y)$ has full row rank, we can use the implicit function theorem again. Partitioning y as $y = (y_1, y_2)$, $y_1 \in \mathbb{R}^{n_z}$, $y_2 \in \mathbb{R}^{n_y - n_z}$, y_2 can be represented by a differentiable function of (x_2, y_2): $y_1 = \phi(x_2, y_2)$ in a neighborhood of (x_{20}, y_{20}) such that

$$\frac{\partial h}{\partial x}g\big((\rho(x_2), x_2), (\phi(x_2, y_2), y_2)\big) = 0$$

holds for every pair (x_2, y_2) in this neighborhood. By the same reasoning, since

$$\frac{\partial h}{\partial x}\frac{\partial g}{\partial y}\frac{\partial f}{\partial z} \neq 0,$$

z can be represented as a differentiable function k of $(x, x(t-1), y, y(t-1))$ in some neighborhood of (x_0, x_0, y_0, y_0) and the three aforementioned neighborhoods intersect.

We now perform the coordinate change as before. This coordinate change makes the new variables independent of each other and thus the linearization can be done.

Define the coordinate change by

$$u=\begin{bmatrix}u_1\\u_2\\u_3\\u_4\end{bmatrix}=U(x,y)=\begin{bmatrix}x_1-\rho(x_2)\\x_2\\y_1-\phi(x_2,y_2)\\y_2\end{bmatrix}, \tag{16}$$

so we will have $u_1=0,\ u_3=0$ as new constraints. Under the coordinate transformation (16), the stationary solution (x_0,y_0) is given by $u_0=U(x_0,y_0)$. From (16) we obtain

$$\begin{bmatrix}x_1\\x_2\\y_1\\y_2\end{bmatrix}=Q(u)=\begin{bmatrix}u_1+\rho(u_2)\\u_2\\u_3+\phi(u_2,u_4)\\u_4\end{bmatrix},$$

and

$$T=\frac{\partial Q}{\partial u}=\begin{bmatrix}I_d & \dfrac{\partial\rho(u_2)}{\partial u_2} & 0 & 0\\ 0 & I_{n-d} & 0 & 0\\ 0 & \dfrac{\partial\phi(u_2,u_4)}{\partial u_2} & I_d & \dfrac{\partial\phi(u_2,u_4)}{\partial u_4}\\ 0 & 0 & 0 & I_{m-d}\end{bmatrix}.$$

Note that T is still nonsingular here and $u(t-1),(x(t-1),y(t-1))$ have the same relationship as defined above. In the new variables, (15) becomes

$$\begin{aligned}
Tu'&=\begin{bmatrix}g(Q(u))\\ f\big(Q(u),Q\big(u(t-1)\big),k\big(Q(u),Q\big(u(t-1)\big)\big)\big)\end{bmatrix},\\
0&=u_1,\\
0&=u_3,
\end{aligned}\tag{17}$$

which contains the reduced DDE problem for (u_2,u_4).

By using the fact that $u_1=u_3=0$ when $x_1=\rho(x_2),\ y_1=\phi(x_2,y_2)$, (17) can be written as

$$\frac{\mathrm{d}\rho(u_2)}{\mathrm{d}t}=\frac{\partial\rho(u_2)}{\partial u_2}u_2'=g_1\big(Q(u)\big), \tag{18a}$$

$$u_2'=g_2\big(Q(u)\big), \tag{18b}$$

$$\begin{aligned}\frac{\mathrm{d}\phi(u_2,u_4)}{\mathrm{d}t}&=\frac{\partial\phi(u_2,u_4)}{\partial u_2}u_2'+\frac{\partial\phi(u_2,u_4)}{\partial u_4}u_4'\\ &=f_1\big(Q(u),Q\big(u(t-1)\big),k\big(Q(u),Q\big(u(t-1)\big)\big)\big),\end{aligned} \tag{18c}$$

$$u_4'=f_2\big(Q(u),Q\big(u(t-1)\big),k\big(Q(u),Q\big(u(t-1)\big)\big)\big), \tag{18d}$$

$$0=u_1, \tag{18e}$$

$$0 = u_3, \tag{18f}$$

when (18b) and (18d)–(18f) are satisfied, (18a), (18c) are satisfied automatically. By the same reasoning as in Section 2.2, (18a) can be obtained from (18b), (18e). For (18c), when $u_1 = 0$ and $u_3 = 0$, we can write the implicit constraint as

$$\frac{\partial h}{\partial x} g(x, y) = \frac{\partial h}{\partial x} g\big(Q(u)\big) = \frac{\partial h}{\partial x} g\big(\rho(u_2), u_2, \phi(u_2, u_4), u_4\big).$$

Taking the derivative in time of the above equation, we get

$$\begin{aligned}
&\frac{\partial h}{\partial x}\left(\frac{\partial g}{\partial x_1} g_1 + \frac{\partial g}{\partial x_2} g_2 + \frac{\partial g}{\partial y_1} f_1 \frac{\partial f}{\partial y_2} f_2\right) \\
&\quad = \frac{\partial h}{\partial x}\left(\frac{\partial g}{\partial x_1}\frac{\mathrm{d}\rho(u_2)}{\mathrm{d}t} + \frac{\partial g}{\partial x_2} u_2' + \frac{\partial g}{\partial y_1}\frac{\mathrm{d}\phi(u_2, u_4)}{\mathrm{d}t} + \frac{\partial g}{\partial y_2} u_4'\right),
\end{aligned}$$

which gives (18c) when (18b), (18d) are satisfied.

Linearizing (17) yields

$$\begin{aligned}
Tu' &= \begin{bmatrix} \dfrac{\partial g}{\partial(x, y)} T \\ \dfrac{\partial f}{\partial(x, y)} T + \dfrac{\partial f}{\partial z}\dfrac{\partial k}{\partial(x, y)} T \end{bmatrix} u \\
&\quad + \begin{bmatrix} 0 \\ \dfrac{\partial f}{\partial(x(t-1), y(t-1))} T + \dfrac{\partial f}{\partial z}\dfrac{\partial k}{\partial(x(t-1), y(t-1))} T \end{bmatrix} u(t-1),
\end{aligned} \tag{19a}$$

$$0 = u_1, \tag{19b}$$

$$0 = u_3. \tag{19c}$$

The direct linearization of (15) is given by

$$x' = \frac{\partial g}{\partial x} x + \frac{\partial g}{\partial y} y, \tag{20a}$$

$$y' = \frac{\partial f}{\partial x} x + \frac{\partial f}{\partial x(t-1)} x(t-1) + \frac{\partial f}{\partial y} y + \frac{\partial f}{\partial y(t-1)} y(t-1) \frac{\partial f}{\partial z} z, \tag{20b}$$

$$0 = \frac{\partial h}{\partial x} x. \tag{20c}$$

Differentiating the constraint (20c) twice and noting that

$$\frac{\partial h}{\partial x}\frac{\partial g}{\partial y}\frac{\partial f}{\partial z} \neq 0,$$

we can solve for z. Inserting z back into (20a), we note that the equations involve only x and y,

$$x' = \frac{\partial g}{\partial x} x + \frac{\partial g}{\partial y} y, \tag{21a}$$

$$
\begin{aligned}
y' = {} & \left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial z}\left(\frac{\partial h}{\partial x}\frac{\partial g}{\partial y}\frac{\partial f}{\partial z}\right)^{-1}\left(\frac{\partial h}{\partial x}\frac{\partial g}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial h}{\partial x}\frac{\partial g}{\partial y}\frac{\partial f}{\partial x}\right)\right)x \\
& + \left(\frac{\partial f}{\partial x(t-1)} - \frac{\partial f}{\partial z}\left(\frac{\partial h}{\partial x}\frac{\partial g}{\partial y}\frac{\partial f}{\partial z}\right)^{-1}\frac{\partial h}{\partial x}\frac{\partial g}{\partial y}\frac{\partial f}{\partial x(t-1)}\right)x(t-1) \\
& + \left(\frac{\partial f}{\partial y} - \frac{\partial f}{\partial z}\left(\frac{\partial h}{\partial x}\frac{\partial g}{\partial y}\frac{\partial f}{\partial z}\right)^{-1}\left(\frac{\partial h}{\partial x}\frac{\partial g}{\partial x}\frac{\partial g}{\partial y} + \frac{\partial h}{\partial x}\frac{\partial g}{\partial y}\frac{\partial f}{\partial y}\right)\right)y \\
& + \left(\frac{\partial f}{\partial y(t-1)} - \frac{\partial f}{\partial z}\left(\frac{\partial h}{\partial x}\frac{\partial g}{\partial y}\frac{\partial f}{\partial z}\right)^{-1}\frac{\partial h}{\partial x}\frac{\partial g}{\partial y}\frac{\partial f}{\partial y(t-1)}\right)y(t-1), \qquad (21b)\\
0 = {} & \frac{\partial h}{\partial x}x, \qquad (21c)\\
0 = {} & \frac{\partial h}{\partial x}\left(\frac{\partial g}{\partial x}x + \frac{\partial g}{\partial y}y\right), \qquad (21d)
\end{aligned}
$$

where (21d) is added as an implicit constraint which will be shown to be equivalent to (19c). (21) in fact contains a DDE subproblem for x_2, y_2. Now let

$$
\begin{bmatrix} x \\ y \end{bmatrix} = T(u_0)u, \qquad \begin{bmatrix} x(t-1) \\ y(t-1) \end{bmatrix} = T(u_0)u(t-1),
$$

so that (21) can be written as

$$
Tu' = \begin{bmatrix} \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \\ F_1 & F_2 \end{bmatrix} Tu + \begin{bmatrix} 0 & 0 \\ F_3 & F_4 \end{bmatrix} Tu(t-1), \qquad (22a)
$$

$$
0 = \frac{\partial h}{\partial x}\begin{bmatrix} I_{n_z} & \dfrac{\partial \rho(u_2)}{\partial u_2} \\ 0 & I_{n_y-n_z} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \qquad (22b)
$$

$$
\begin{aligned}
0 = {} & \frac{\partial h}{\partial x}\frac{\partial g}{\partial x}\left.\begin{bmatrix} I_d & \dfrac{\partial \rho(u_2)}{\partial u_2} \\ 0 & I_{n-d} \end{bmatrix}\right|_{u_0} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
& + \frac{\partial h}{\partial x}\frac{\partial g}{\partial y}\begin{bmatrix} \dfrac{\partial \phi(u_2,u_4)}{\partial u_2}u_2 + u_3 + \dfrac{\partial \phi(u_2,u_4)}{\partial u_4}u_4 \\ u_4 \end{bmatrix}, \qquad (22c)
\end{aligned}
$$

where F_1, F_2, F_3, F_4 are the matrices multiplying x, y, $x(t-1)$, $y(t-1)$ in (21), respectively. By the same reasoning as in Section 2.2, (22b) and (19b) are equivalent. The equivalence between (22c) and (19c) can be shown by the following. Since we have that in Θ

$$
\frac{\partial h}{\partial x}g\big(\rho(u_2), u_2, \phi(u_2,u_4), u_4\big) = 0
$$

holds for every pair (u_2, u_4), thus

$$
\frac{\mathrm{d}}{\mathrm{d}u_1}\left(\frac{\partial h}{\partial x}g\big(\rho(u_2), u_2, \phi(u_2,u_4), u_4\big)\right) = \frac{\mathrm{d}}{\mathrm{d}u_2}\left(\frac{\partial h}{\partial x}g\big(\rho(u_2), u_2, \phi(u_2,u_4), u_4\big)\right) = 0,
$$

which at the fixed point gives

$$\frac{\partial h}{\partial x}\left(\frac{\partial g}{\partial x_1}\frac{\partial \rho(u_2)}{\partial u_2}+\frac{\partial g}{\partial x_2}+\frac{\partial g}{\partial y_1}\frac{\partial \phi(u_2,u_4)}{\partial u_2}\right)=0,$$

$$\frac{\partial h}{\partial x}\left(\frac{\partial g}{\partial y_1}\frac{\partial \phi(u_2,u_4)}{\partial u_4}+\frac{\partial g}{\partial y_2}\right)=0.$$

Using these equations, we can simplify (22c) to

$$\frac{\partial h}{\partial x}\left(\frac{\partial g}{\partial x_1}u_1+\frac{\partial g}{\partial y_1}u_3\right)=0.$$

Since we already proved that (22b) is equivalent to (19b), it follows that $u_1=0$. Noting that

$$\frac{\partial h}{\partial x}\frac{\partial g}{\partial y_1}\neq 0,$$

we obtain that $u_3=0$ is equivalent to (22c).

Now comparing (22a) with (19b), we see that to show that the linearizations (19) and (22) are the same, it is enough to prove

$$[F_1\ F_2]=\frac{\partial f}{\partial(x,y)}+\frac{\partial f}{\partial z}\frac{\partial k}{\partial(x,y)},$$

$$[F_3\ F_4]=\frac{\partial f}{\partial(x(t-1),y(t-1))}+\frac{\partial f}{\partial z}\frac{\partial k}{\partial(x(t-1),y(t-1))},$$

at the fixed points. Since

$$\frac{\partial f}{\partial(x,y)}=\left[\frac{\partial f}{\partial x}\ \frac{\partial f}{\partial y}\right],$$

$$\frac{\partial f}{\partial(x(t-1),y(t-1))}=\left[\frac{\partial f}{\partial x(t-1)}\ \frac{\partial f}{\partial y(t-1)}\right],$$

$$\frac{\partial f}{\partial z}\frac{\partial k}{\partial(x,y)}=\left[\frac{\partial f}{\partial z}\frac{\partial k}{\partial x}\ \frac{\partial f}{\partial z}\frac{\partial k}{\partial y}\right],$$

$$\frac{\partial f}{\partial z}\frac{\partial k}{\partial(x(t-1),y(t-1))}=\left[\frac{\partial f}{\partial z}\frac{\partial k}{\partial x(t-1)}\ \frac{\partial f}{\partial z}\frac{\partial k}{\partial y(t-1)}\right],$$

we just need to prove that at the fixed points

$$\frac{\partial f}{\partial z}\frac{\partial k}{\partial x}+\frac{\partial f}{\partial z}\left(\frac{\partial h}{\partial x}\frac{\partial g}{\partial y}\frac{\partial f}{\partial z}\right)^{-1}\left(\frac{\partial h}{\partial x}\frac{\partial g}{\partial x}\frac{\partial g}{\partial x}+\frac{\partial h}{\partial x}\frac{\partial g}{\partial y}\frac{\partial f}{\partial x}\right)=0,\tag{23a}$$

$$\frac{\partial f}{\partial z}\frac{\partial k}{\partial x(t-1)}+\frac{\partial f}{\partial z}\left(\frac{\partial h}{\partial x}\frac{\partial g}{\partial y}\frac{\partial f}{\partial z}\right)^{-1}\frac{\partial h}{\partial x}\frac{\partial g}{\partial y}\frac{\partial f}{\partial x(t-1)}=0,\tag{23b}$$

$$\frac{\partial f}{\partial z}\frac{\partial k}{\partial y}+\frac{\partial f}{\partial z}\left(\frac{\partial h}{\partial x}\frac{\partial g}{\partial y}\frac{\partial f}{\partial z}\right)^{-1}\left(\frac{\partial h}{\partial x}\frac{\partial g}{\partial x}\frac{\partial g}{\partial y}+\frac{\partial h}{\partial x}\frac{\partial g}{\partial y}\frac{\partial f}{\partial y}\right)=0,\tag{23c}$$

$$\frac{\partial f}{\partial z}\frac{\partial k}{\partial y(t-1)}+\frac{\partial f}{\partial z}\left(\frac{\partial h}{\partial x}\frac{\partial g}{\partial y}\frac{\partial f}{\partial z}\right)^{-1}\frac{\partial h}{\partial x}\frac{\partial g}{\partial y}\frac{\partial f}{\partial y(t-1)}=0. \tag{23d}$$

We give the proof for (23a) and (23b), noting that (23c) and (23d) can be proved similarly.

Since differentiating the constraint $h(x)=0$ twice gives

$$\frac{\partial h}{\partial x}\frac{\partial g}{\partial x}g(x,y)+\frac{\partial h}{\partial x}\frac{\partial g}{\partial y}f\big(x,x(t-1),y,y(t-1),z\big)=0,$$

and as mentioned before

$$\frac{\partial h}{\partial x}\frac{\partial g}{\partial y}\frac{\partial f}{\partial z}\neq 0,$$

in a neighborhood Θ of (x_0,x_0,y_0,y_0), z can be represented as a differentiable function k of $(x, x(t-1),y,y(t-1))$ and the following equation holds for every pencil $(x,x(t-1),y,y(t-1))$ in Θ:

$$F(x,y)=\frac{\partial h}{\partial x}\frac{\partial g}{\partial x}g(x,y)+\frac{\partial h}{\partial x}\frac{\partial g}{\partial y}f\big(x,x(t-1),y,y(t-1),k\big(x,x(t-1),y,y(t-1)\big)\big)=0.$$

So we have

$$\frac{\mathrm{d}F}{\mathrm{d}x}=\frac{\mathrm{d}F}{\mathrm{d}y}=\frac{\mathrm{d}F}{\mathrm{d}x(t-1)}=\frac{\mathrm{d}F}{\mathrm{d}y(t-1)}=0$$

in Θ and at the fixed points

$$\frac{\mathrm{d}F}{\mathrm{d}x}=\frac{\partial h}{\partial x}\frac{\partial g}{\partial x}\frac{\partial g}{\partial x}+\frac{\partial h}{\partial x}\frac{\partial g}{\partial y}\frac{\partial f}{\partial x}+\frac{\partial h}{\partial x}\frac{\partial g}{\partial y}\frac{\partial f}{\partial z}\frac{\partial k}{\partial x}=0,$$

$$\frac{\mathrm{d}F}{\mathrm{d}x(t-1)}=\frac{\partial h}{\partial x}\frac{\partial g}{\partial y}\frac{\partial f}{\partial x(t-1)}+\frac{\partial h}{\partial x}\frac{\partial g}{\partial y}\frac{\partial f}{\partial z}\frac{\partial k}{\partial x(t-1)}=0,$$

from which $\partial k/\partial x$ and $\partial k/\partial x(t-1)$ can be obtained:

$$\frac{\partial k}{\partial x}=-\left(\frac{\partial h}{\partial x}\frac{\partial g}{\partial y}\frac{\partial f}{\partial z}\right)^{-1}\left(\frac{\partial h}{\partial x}\frac{\partial g}{\partial x}\frac{\partial g}{\partial x}+\frac{\partial h}{\partial x}\frac{\partial g}{\partial y}\frac{\partial f}{\partial x}\right),$$

$$\frac{\partial k}{\partial x(t-1)}=-\left(\frac{\partial h}{\partial x}\frac{\partial g}{\partial y}\frac{\partial f}{\partial z}\right)^{-1}\frac{\partial h}{\partial x}\frac{\partial g}{\partial y}\frac{\partial f}{\partial x(t-1)}.$$

Thus we see that (23) holds, and the equivalence between the two linearizations (19) and (22) is established.

3. Hessenberg DDAEs of neutral type

We consider the index-one problem given in [1] which is of neutral type

$$\begin{aligned} x' &= f\big(x,x(t-1),y,(t-1)\big),\\ 0 &= g\big(x,x(t-1),y,y(t-1)\big), \end{aligned} \tag{24}$$

where $f:U_1\to V_1$, $g:U_2\to V_2$, $U_1\subseteq\mathbb{R}^{n_x}\times\mathbb{R}^{n_x}\times\mathbb{R}^{n_y}\times\mathbb{R}^{n_y}$, $U_2\subseteq\mathbb{R}^{n_x}\times\mathbb{R}^{n_x}\times\mathbb{R}^{n_y}\times\mathbb{R}^{n_y}$, $V_1\subseteq\mathbb{R}^{n_x}$, $V_2\subseteq\mathbb{R}^{n_y}$ and g_y nonsingular.

Let (x_0, y_0) be a fixed point of (24). We have

$$\begin{aligned} 0 &= f(x_0, x_0, y_0, y_0), \\ 0 &= g(x_0, x_0, y_0, y_0). \end{aligned} \tag{25}$$

By the implicit function theorem, there exists a neighborhood Θ of (x_0, x_0, y_0, y_0) and a differentiable function $\rho(x, x(t-1), y(t-1))$ such that in Θ, we have

$$y = \rho\big(x, x(t-1), y(t-1)\big), \tag{26}$$

and

$$g\big(x, x(t-1), \rho\big(x, x(t-1), y(t-1)\big), y(t-1)\big) = 0$$

holds for any $(x, x(t-1), y(t-1))$ in Θ.

Thus in Θ the constraint can be satisfied automatically and we obtain the reduced state space form DDE of free variable x as

$$x' = f\big(x, x(t-1), \rho\big(x, x(t-1), y(t-1)\big), y(t-1)\big), \tag{27}$$

where

$$y(t-1) = \rho\big(x(t-1), x(t-2), y(t-2)\big)$$

and (27) can be understood recursively. It is obvious from (27) that (24) is of neutral type since $x'(t),\ t \geqslant 0$, depends directly on $x(t),\ -1 \leqslant t \leqslant 0$, and hence the discontinuity of $x(t)$ in $-1 \leqslant t \leqslant 0$ will not be smoothed out.

Let $\rho = \rho(w_1, w_2, w_3)$. The direct linearization of (27) along the stationary solution, ignoring the dependence of $y(t-1)$ on $x(t-1)$, gives

$$\begin{aligned} x' = {} & \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial \rho}{\partial w_1}\right)x + \left(\frac{\partial f}{\partial x(t-1)} + \frac{\partial f}{\partial y}\frac{\partial \rho}{\partial w_2}\right)x(t-1) \\ & + \left(\frac{\partial f}{\partial y}\frac{\partial \rho}{\partial w_3} + \frac{\partial f}{\partial y(t-1)}\right)y(t-1), \end{aligned} \tag{28}$$

where

$$y(t-1) = \frac{\partial \rho}{\partial w_1}x(t-1) + \frac{\partial \rho}{\partial w_2}x(t-2) + \frac{\partial \rho}{\partial w_3}y(t-2),$$

and $y(t-2)$ is considered as having a similar representation as $y(t-1)$ and so on. If $v-1 \leqslant t \leqslant v$, where v is some integer, then we obtain

$$\begin{aligned} y(t-1) = {} & \frac{\partial \rho}{\partial w_1}x(t-1) + \sum_{i=2}^{v}\left(\left(\frac{\partial \rho}{\partial w_3}\right)^{i-2}\frac{\partial \rho}{\partial w_2} + \left(\frac{\partial \rho}{\partial w_3}\right)^{i-1}\frac{\partial \rho}{\partial w_1}\right)x(t-i) \\ & + \left(\frac{\partial \rho}{\partial w_3}\right)^{v} y(t-v). \end{aligned} \tag{29}$$

Inserting (29) into (28), we get the final form of the linearization of (27) as

$$x' = \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial \rho}{\partial w_1}\right)x + \left(\frac{\partial f}{\partial x(t-1)} + \frac{\partial f}{\partial y}\frac{\partial \rho}{\partial w_2} + \frac{\partial f}{\partial y}\frac{\partial \rho}{\partial w_3}\frac{\partial \rho}{\partial w_1} + \frac{\partial f}{\partial y(t-1)}\frac{\partial \rho}{\partial w_1}\right)x(t-1)$$
$$+ \left(\frac{\partial f}{\partial y}\frac{\partial \rho}{\partial w_3} + \frac{\partial f}{\partial y(t-1)}\right)\sum_{i=2}^{v}\left(\left(\frac{\partial \rho}{\partial w_3}\right)^{i-2}\frac{\partial \rho}{\partial w_2} + \left(\frac{\partial \rho}{\partial w_3}\right)^{i-1}\frac{\partial \rho}{\partial w_1}\right)x(t-i)$$
$$+ \left(\frac{\partial f}{\partial y}\frac{\partial \rho}{\partial w_2} + \frac{\partial f}{\partial y(t-1)}\right)\left(\frac{\partial \rho}{\partial w_3}\right)^{v} y(t-v). \tag{30}$$

Eq. (30) is equivalent to the direct linearization of the equation obtained by expanding out (27) thoroughly in the variables $x(t-1)$, $x(t-2), \ldots, x(t-v)$, $y(t-v)$ for $v-1 \leqslant t \leqslant v$ using the recursive representation of $y(t-i)$. This can be shown by direct computation. We omit the proof here.

The direct linearization of (24) is given by

$$x' = \frac{\partial f}{\partial x}x + \frac{\partial f}{\partial x(t-1)}x(t-1) + \frac{\partial f}{\partial y}y + \frac{\partial f}{\partial y(t-1)}y(t-1), \tag{31a}$$

$$0 = \frac{\partial g}{\partial x}x + \frac{\partial g}{\partial x(t-1)}x(t-1) + \frac{\partial g}{\partial y}y + \frac{\partial g}{\partial y(t-1)}y(t-1). \tag{31b}$$

From (31b), we obtain

$$y = -\left(\frac{\partial g}{\partial y}\right)^{-1}\left(\frac{\partial g}{\partial x}x + \frac{\partial g}{\partial x(t-1)}x(t-1) + \frac{\partial g}{\partial y(t-1)}y(t-1)\right), \tag{32}$$

which can be inserted back into (31a) to obtain

$$x' = \left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\left(\frac{\partial g}{\partial y}\right)^{-1}\frac{\partial g}{\partial x}\right)x + \left(\frac{\partial f}{\partial x(t-1)} - \frac{\partial f}{\partial y}\left(\frac{\partial g}{\partial y}\right)^{-1}\frac{\partial g}{\partial x(t-1)}\right)x(t-1)$$
$$+ \left(\frac{\partial f}{\partial y(t-1)} - \frac{\partial f}{\partial y}\left(\frac{\partial g}{\partial y}\right)^{-1}\frac{\partial g}{\partial y(t-1)}\right)y(t-1). \tag{33}$$

Since we have $0 = g(x, x(t-1), y, y(t-1))$ for every $(x, x(t-1), y(t-1))$ in Θ, where x, $x(t-1)$, $y(t-1)$ are considered as independent variables, we obtain

$$\begin{aligned}
\frac{\mathrm{d}g}{\mathrm{d}x} &= \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}\frac{\partial \rho}{\partial w_1},\\
\frac{\mathrm{d}g}{\mathrm{d}x(t-1)} &= \frac{\partial g}{\partial x(t-1)} + \frac{\partial g}{\partial y}\frac{\partial \rho}{\partial w_2},\\
\frac{\mathrm{d}g}{\mathrm{d}y(t-1)} &= \frac{\partial g}{\partial y(t-1)} + \frac{\partial g}{\partial y}\frac{\partial \rho}{\partial w_3}.
\end{aligned} \tag{34}$$

Thus (33) is equivalent to

$$x' = \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial \rho}{\partial w_1}\right)x + \left(\frac{\partial f}{\partial x(t-1)} + \frac{\partial f}{\partial y}\frac{\partial \rho}{\partial w_2}\right)x(t-1)$$
$$+ \left(\frac{\partial f}{\partial y(t-1)} + \frac{\partial f}{\partial y}\frac{\partial \rho}{\partial w_3}\right)y(t-1). \tag{35}$$

Linearization (35) is just (28), which is equivalent to (30). Eq. (30) is equivalent to the linearization of (27) obtained by expanding out thoroughly in the variables $x(t-i)$, $1 \leqslant i \leqslant v$, and $y(t-v)$ for $v-1 \leqslant t \leqslant v$. Thus we have shown the equivalence between the direct linearization of the original neutral delay problem and the linearization of its reduced form.

Proofs of validity of the direct linearizations of the index-two and index-three neutral problems are much more complicated using this method. We expect a differential geometric approach like [16] will handle such problems more easily.

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