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CANONICAL FORMS AND SOLVABLE SINGULAR SYSTEMS OF DIFFERENTIAL EQUATIONS*

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Abstract. In this paper we investigate the relationship between solvability and the existence of canonical forms for the linear system of differential equations $E(t)x'(t)+F(t)x(t)=f(t)$. We show that if E, F are analytic on the interval $[0\ \ T]$, then the differential equation is solvable if and only if it can be put into a certain canonical form. We give examples to show that this is not true if E, F are only differentiable.

1. Introduction.

Linear systems of differential equations of the form

$$E(t)x'(t)+F(t)x(t)=f(t) \tag{1}$$

with $E(t)$ a singular $n\times n$ matrix occur in a wide variety of circuit and control applications. Many of these applications are described in some detail in [3], [4], see also [2]. The constant coefficient case is now fairly well understood. However, the theory for the time varying case is still incomplete. This note has several purposes. One is to clear up some of the misconceptions and confusion in the current literature. A second is to give some new results.

We shall say (1) is *analytically solvable* on the interval $[0\ \ T]$ if for any sufficiently smooth (C^n will do) f there exist solutions to (1), and solutions when they exist, are defined on all of $[0\ \ T]$ and are uniquely determined by their value at any $t_0\in[0\ \ T]$. It is useful to note that a system fails to be analytically solvable if it has any *turning points* in $[0\ \ T]$ (where by turning point we mean a point where the dimension of the manifold of solutions changes), since at these points solutions fail either to exist or to be unique.

If (1) is in the form

$$y_1'+C(t)y_1=f_1, \tag{2a}$$

$$N(t)y_2'+y_2=f_2, \tag{2b}$$

where $N(t)$ is nilpotent and lower (or upper) triangular, the system is said to be in *standard canonical form*, SCF [7]. If, in addition, N is constant, then the system is in *strong standard canonical form*, SSCF. The SSCF is the one considered in the work of Petzold and Gear [8], [9], [11].

We shall consider transformations of the form

$$x=Q(t)y, \text{ and left multiplication of the equation by } P(t), \tag{3}$$

where P, Q are invertible on $[0\ \ T]$ and are as smooth as E, F. Clearly, if (1) can be put into SCF, it is analytically solvable. Recently some authors have suggested that analytic solvability implies SSCF except at a finite number of isolated points. In § 2 we shall give a series of examples that show this is not the case unless the matrices $E(t)$, $F(t)$ in (1) are analytic functions of t. In § 3 we prove that if E, F are analytic

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on $[0\ \ T]$, then analytic solvability implies (1) can be transformed by (3) to SCF everywhere on $[0\ \ T]$.

2. Examples.

Example 1. Let $\eta(t)$ be an infinitely differentiable function defined on $[0\ \ T]$ so that $\eta=0$ on $[2^{-k-1}\ \ 2^{-k}]$ for k even and $\eta>0$ otherwise. Consider the system

$$\begin{bmatrix} 0 & 0 \\ \eta & 0 \end{bmatrix}\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}+\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \tag{4}$$

which has the solution

$$x_1=f_1, \tag{5a}$$

$$x_2=f_2-\eta f_1'. \tag{5b}$$

Clearly (4) is solvable on $[0\ \ 1]$ and already in SCF. However any P, Q putting (4) into SSCF will have discontinuities at $\{2^{-k}: k \text{ even}\}\cup\{0\}$.

This example is also interesting from another point of view. Suppose $\eta(t)$ is analytic on $[0\ \ T]$, $\eta(t^*)=0$, $t^*\in[0\ \ T]$ and nonzero on $[0\ \ T]\backslash\{t^*\}$. Then (4) is transformable to SSCF on any closed subinterval of $[0\ \ T]\backslash\{t^*\}$. The point t^* is not a turning point as we defined it in § 1. However, the system (4) does have, in some sense, a structure change at t^* since the coefficient of x' changes rank and index at t^*.

Example 2. Let

$$N(t)=t^3\begin{bmatrix} \sin(t^{-1}) \\ \cos(t^{-1}) \end{bmatrix}[\cos(t^{-1}), -\sin(t^{-1})], \qquad N(0)=0. \tag{6}$$

Note that $N'(0)=0$ and $N^2\equiv 0$. Thus there is an interval containing zero so that

$$Nx'+x=f \tag{7}$$

is solvable on that interval [5]. However, if $\psi(t)$ is a vector valued function so that $N\psi\equiv 0$, $\psi(0)\neq 0$, then $\psi(t)$ is a multiple of $[\sin(t^{-1}), \cos(t^{-1})]$ and hence is discontinuous at $t=0$. In particular any P, Q putting (6), (7) into SCF must be discontinuous at zero.

A slight modification of Example 2, along the lines of Example 1, can be used to construct a solvable system such that any P, Q putting the system into SCF would have an infinite number of singularities in a finite interval.

3. Analytic coefficients. The essential problem in Example 2 is that if rank $(A(t))\leqq r<n$ for all t and A is $n\times n$, then there need not exist *any* piecewise smooth, nonzero vectors $\psi(t)$ so that $A\psi\equiv 0$. However, it is a not generally known, and nontrivial fact, that such a ψ exists if A is analytic. The version of the result we shall need is the following theorem from [13, p. 335].

THEOREM 1. *If $A(t)$ is real analytic on $[0\ \ T]$ and $r\geqq \operatorname{rank}(A(t))$ for all t, then there exists real analytic $P(t)$, $Q(t)$ so that*

$$PAQ=\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{8}$$

and A_1 is $r\times r$.

An infinite dimensional version of Theorem 1 appears in [1, Thm. 2.2]. See also [10].

We are now in a position to prove the main result of this section.

THEOREM 2. *If E, F are analytic on $[0\ \ T]$, and (1) is analytically solvable, then there exist analytic P, Q so that the transformations (3) put (1) into* SCF.

The key to proving Theorem 2 is to first show that while E may have variable rank, solvability forces E to be always singular or always nonsingular.

LEMMA 1. *If (1) is analytically solvable on $[0\ \ T]$, then E is either always singular or always nonsingular on $[0\ \ T]$.*

Proof of Lemma 1. Suppose for purposes of contradiction that $E(t_0)$ is nonsingular and $E(t_1)$ is singular. Then for any f, there are n linearly independent solutions of (1) at t_0. Let ψ be a vector so that $\psi^T E(t_1) = 0$. Now multiply (1) by ψ^T and evaluate at t_1 to observe that $\psi^T F(t_1)x(t_1) = \psi^T f$. The case $f = \psi$ implies $\psi^T F(t_1) \neq 0$. Hence all solutions for $f = 0$ satisfy $\psi^T F(t_1)x(t_1) = 0$ and are not linearly independent at t_1 which contradicts analytic solvability. □

Proof of Theorem 2. Suppose Theorem 2 is not true and that E, F are analytic on $[0\ \ T]$, (1) is analytically solvable, but it is not possible to transform to SCF and E, F give a counterexample of minimum possible dimension n. Clearly E is singular and $n > 1$. By Lemma 1, E is always singular on $[0\ \ T]$. Let r be such that rank $E \leqq r < n$ on $[0\ \ T]$. Let P be such that

$$PE = \begin{bmatrix} E_1 & E_2 \\ 0 & 0 \end{bmatrix} \tag{9}$$

where E_1 is $r \times r$ and P is analytic on $[0\ \ T]$. Such a P exists by Theorem 1. Multiplying (1) by P gives the still analytically solvable system

$$\begin{bmatrix} E_1 & E_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} + \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}. \tag{10}$$

But $[F_{21}\ \ F_{22}]$ has full rank $n - r$ on $[0\ \ T]$, since if it did not, there would exist a t_0, $\psi \neq 0$, such that $\psi^T[F_{21}(t)\ \ F_{22}(t)] = 0$. But then

$$[0, \psi^T]\begin{bmatrix} f_1(t_0) \\ f_2(t_0) \end{bmatrix} = \psi^T f_2(t_0) = 0,$$

which contradicts the fact that f can be an arbitrary function. Now there exists an invertible analytic Q on $[0\ \ T]$ so that $[F_{21}\ \ F_{22}]Q = [0\ \ G_{22}]$ where G_{22} is invertible. Note this follows as (9) by using

$$Q^T\begin{bmatrix} F_{21}^T & 0 \\ F_{22}^T & 0 \end{bmatrix} = \begin{bmatrix} G_{22}^T & 0 \\ 0 & 0 \end{bmatrix}.$$

Making the change of variable $x = Qy$ turns (10) into

$$\begin{bmatrix} \hat{E}_1 & \hat{E}_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} + \begin{bmatrix} \hat{F}_{11} & \hat{F}_{12} \\ 0 & G_{22} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \tag{11}$$

where $[\hat{E}_1\ \hat{E}_2] = [E_1\ \ E_2]Q$, $[\hat{F}_{11}\ \hat{F}_{12}] = [F_{11}\ \ F_{12}]Q + [E_1\ \ E_2]Q'$. But (11) is equivalent to solving

$$\hat{E}_1 y_1' + \hat{F}_{11} y_1 = f_1 - \hat{E}_2(G_{22}^{-1} f_2)' - \hat{F}_{12} G_{22}^{-1} f_2, \tag{12}$$

or

$$\hat{E}_1 y_1' + \hat{F}_{11} y_1 = \hat{f} \tag{13}$$

for arbitrary smooth $\hat{f}$. Thus (13) is also an analytically solvable system. Since it has lower dimension than n, by assumption, there exists analytic $R_1(t)$, $R_2(t)$ that puts

(13) into (upper triangular) SCF. Letting

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} R_2 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

and multiplying by

$$\begin{bmatrix} R_1 & 0 \\ 0 & I \end{bmatrix}$$

changes (11) to

$$\left[\begin{array}{cc|c} I & 0 & \tilde{E}_{13} \\ 0 & N(t) & \tilde{E}_{23} \\ \hline 0 & 0 & 0 \end{array}\right] \begin{bmatrix} z'_{11} \\ z'_{12} \\ z'_2 \end{bmatrix} + \left[\begin{array}{cc|c} C(t) & 0 & \tilde{F}_{13} \\ 0 & I & \tilde{F}_{23} \\ \hline 0 & 0 & G_{22} \end{array}\right] \begin{bmatrix} z_{11} \\ z_{12} \\ z_2 \end{bmatrix} = \begin{bmatrix} f_{11} \\ f_{12} \\ \tilde{f}_2 \end{bmatrix}. \tag{14}$$

Now multiply this equation by

$$\begin{bmatrix} I & 0 & 0 \\ 0 & \multicolumn{2}{c}{\begin{bmatrix} I & \tilde{F}_{23} \\ 0 & Q_{22} \end{bmatrix}^{-1}} \\ 0 & & \end{bmatrix}$$

to yield

$$\left[\begin{array}{c|cc} I & 0 & \tilde{E}_{13} \\ \hline 0 & N(t) & \tilde{E}_{23} \\ 0 & 0 & 0 \end{array}\right] \begin{bmatrix} z'_{11} \\ z'_{12} \\ z'_2 \end{bmatrix} + \begin{bmatrix} C(t) & 0 & \tilde{F}_{13} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} z_{11} \\ z_{12} \\ z_2 \end{bmatrix} = \begin{bmatrix} \bar{f}_{11} \\ \bar{f}_{12} \\ \bar{f}_2 \end{bmatrix}. \tag{15}$$

Now let

$$z = \begin{bmatrix} I & 0 & -\tilde{E}_{13} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} w$$

and multiply by

$$\begin{bmatrix} I & 0 & \tilde{E}'_{13} + C\tilde{E}_{13} - \tilde{F}_{13} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

to get the SCF

$$\left[\begin{array}{c|cc} I & 0 & 0 \\ \hline 0 & N(t) & E_{23} \\ 0 & 0 & 0 \end{array}\right] \begin{bmatrix} w'_1 \\ w'_2 \\ w'_3 \end{bmatrix} + \begin{bmatrix} C(t) & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} w'_1 \\ w'_2 \\ w'_3 \end{bmatrix} = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \end{bmatrix}, \tag{16}$$

which contradicts the assumption that (1) could not be put into SCF. □

4. Comments. An examination of the proof of Theorem 2 shows that the analyticity of E, F was used only in applying Theorem 1 to get analytic P, Q such that (8), (9) hold. Since (8), (9) hold for many matrix functions met in practice, it seems plausible that the nonexistence of the SCF is an exceptional event.

The approach in this paper differs from those of earlier authors, for example, Silverman [12], in that we do not assume $E(t)$ in (1) has constant rank. In particular, we include systems which cannot be put in the form

$$x_1' = A_{11}(t)x_1 + A_{12}(t)x_2 + f_1(x),$$
$$0 = A_{21}(t)x_1 + A_{22}(t)x_2 + f_2(t)$$

by transformations of the form (3).

The proof of Theorem 2 also provides an algorithmic procedure for obtaining the SCF. Starting with $Ex' + Fx = f$, compute P, Q as in (9), (10) to get (11). Now take the subsystem $\hat{E}_1 y_1' + \hat{F}_{11} y_1 = \hat{f}_1$ of (11) and repeat the procedure again to get again a system in the form (11). At each step we work with a smaller subsystem. At some step we arrive at a system in the form (11) with either $\hat{E}_1$ identically zero or always invertible and the procedure terminates.

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